

# TWO DIMENSIONAL SUBSONIC EULER FLOWS PAST A WALL OR A SYMMETRIC BODY\*

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**ABSTRACT.** The existence and uniqueness of two dimensional steady compressible Euler flows past a wall or a symmetric body are established. More precisely, given positive convex horizontal velocity in the upstream, there exists a critical value  $\rho_{cr}$  such that if the incoming density in the upstream is larger than  $\rho_{cr}$ , then there exists a subsonic flow past a wall. Furthermore,  $\rho_{cr}$  is critical in the sense that there is no such subsonic flow if the density of the incoming flow is less than  $\rho_{cr}$ . The subsonic flows possess large vorticity and positive horizontal velocity above the wall except at the corner points on the boundary. Moreover, the existence and uniqueness of a two dimensional subsonic Euler flow past a symmetric body are also obtained when the incoming velocity field is a general small perturbation of a constant velocity field and the density of the incoming flow is larger than a critical value. The asymptotic behavior of the flows is obtained with the aid of some integral estimates for the velocity field and its far field states.

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## 1. INTRODUCTION AND MAIN RESULTS

One of the most important problems in aerodynamics is to study flows past a body. Mathematical investigation for this problem has a long history. When the flow is irrotational, the study on subsonic flows past a body is quite mature. The existence of two dimensional subsonic irrotational flows past a smooth body with small free stream Mach number was obtained by Shiffman [32]. When the free stream Mach number is less than a critical number, Bers [2] proved the existence of two dimensional subsonic irrotational flows around a general body and also showed that the maximum of Mach numbers approaches to one as the free stream Mach number approaches to the critical value. The uniqueness and asymptotic behavior of subsonic irrotational plane flows were studied in [19]. The existence of three dimensional subsonic irrotational flows around a smooth body were established in [20, 11] when the free stream Mach number is less than a critical number. The fine properties of two dimensional smooth subsonic-sonic irrotational flows around a body were investigated in [22]. The existence of weak solutions for subsonic-sonic flows by a compensated compactness method was obtained in [5, 24]. A significant result by Morawetz shows that, in general, smooth transonic flows past a profile are unstable with respect to small perturbations for the profile, see [27, 28, 29]. Hence one has to deal with transonic flows with discontinuities where in general the flows have non-zero vorticity in the subsonic region.

The vorticity in compressible ideal flows is important not only mathematically but also physically. The main purpose of this paper is to investigate the existence and uniqueness of two dimensional subsonic Euler flows with non-zero vorticity past a wall or a symmetric body. As stated in [1, p.12], “Closely related to the flow around a profile is the flow past a wall”. Subsonic Euler flows with non-zero vorticity in a physical domain were first established in [40] where Xie and Xin studied subsonic Euler flows through an infinitely long smooth nozzle. The major difficulty for the steady Euler system with non-zero vorticity is that the Euler system is a hyperbolic-elliptic coupled system for subsonic flows. In [40], a physical boundary condition for the hyperbolic mode is proposed in the upstream of the flows and the stream function formulation is used to solve the hyperbolic mode [40] so that the steady Euler system is reduced into a single second order equation with memory. This approach was generalized to subsonic flows with non-zero vorticities in nozzles in various settings, such as the flows in periodic nozzles and axially symmetric nozzles, the non-isentropic flows, see [3, 12, 13, 6, 18] and references therein. In particular, the subsonic Euler flows with stagnation points and large vorticity in nozzles were studied in [15, 16].

An attempt for the well-posedness theory for subsonic Euler flows in half plane was made in [9] via the stream function formulation. However, as mentioned in [9, Remark

6.1], the absence of stagnation point in the flow region was not obtained in [9] so that the author in [9] failed to get the equivalence between the stream function formulation and the Euler system, in particular, the uniqueness of the solutions of the original Euler system. Furthermore, the crucial techniques in [9] rely on the estimate for elliptic equations which are small perturbations of the Laplace equation in half plane, so even for the reduced problem for the stream function, the existence and uniqueness of the solution for the problem of the stream function were achieved in [9] only when the incoming velocity is a sufficiently smaller perturbation of a small constant state and the incoming density is a large constant. Our aim in this paper is to prove the existence of subsonic Euler flows, in particular, the flows with large vorticity, as long as the density in the upstream is larger than a critical value. We also prove that subsonic flows above a wall do not have stagnation points and the streamlines of the flows have simple topological structure. The region above the wall can be approximated by a sequence of nozzles so that the analysis in [40, 15, 16] for general quasilinear equation with memory term helps solve these approximated problems. However, the estimates in [40, 15, 16] depend on the height of the nozzles, so one of the key issues in this paper is to prove a series of uniform estimates independent of the nozzle height.

Two-dimensional steady isentropic ideal flows are governed by the following Euler system

$$\begin{cases} \partial_{x_1}(\rho u) + \partial_{x_2}(\rho v) = 0, \\ \partial_{x_1}(\rho u^2) + \partial_{x_2}(\rho uv) + \partial_{x_1}p = 0, \\ \partial_{x_1}(\rho uv) + \partial_{x_2}(\rho v^2) + \partial_{x_2}p = 0, \end{cases} \quad (1.1)$$

where  $(u, v)$  is the velocity,  $\rho$  is the density, and  $p = p(\rho)$  is the pressure of the flow. In this paper, for the simplicity of presentation, we consider the polytropic gas for which the equation of state is  $p = \rho^\gamma$  with the constant  $\gamma > 1$  called the adiabatic exponent. The local sound speed and Mach number of the flow are defined to be

$$c(\rho) = \sqrt{p'(\rho)} = \sqrt{\gamma \rho^{\gamma-1}} \text{ and } M = \frac{\sqrt{u^2 + v^2}}{c(\rho)},$$

respectively. The flow is said to be subsonic if  $M < 1$ , and supersonic if  $M > 1$ .

Consider the flow past a wall  $\Gamma = \{(x_1, f(x_1)) : x_1 \in \mathbb{R}\}$ , i.e., we study the solution of (1.1) in  $\Omega$  defined by

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > f(x_1), \quad -\infty < x_1 < +\infty\}. \quad (1.2)$$

$f(x_1)$  is assumed to be a nonnegative continuous function satisfying

$$f(x_1) > 0 \quad \text{for } x_1 \in (0, 1) \quad \text{and} \quad f(x_1) \equiv 0 \quad \text{for } x_1 \in (-\infty, 0] \cup [1, \infty). \quad (1.3)$$

Furthermore, the curve  $\{(x_1, f(x_1)) : x_1 \in [0, 1]\}$  is a  $C^{1,\alpha}$  smooth curve.

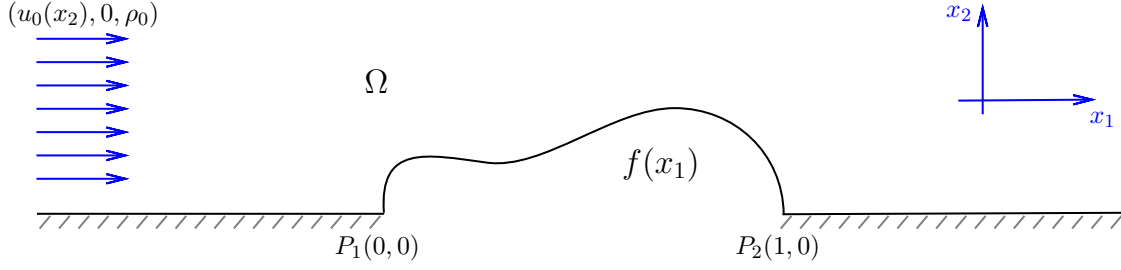


FIGURE 1. Subsonic flows past a wall

The solid wall  $\Gamma$  is assumed to be impermeable and thus,

$$(u, v) \cdot \vec{n} = 0 \quad \text{on } \Gamma, \quad (1.4)$$

where  $\vec{n}$  is the unit outward normal to the boundary  $\Gamma$ . Furthermore, the flow velocity is prescribed in the upstream as

$$(u(x_1, x_2), v(x_1, x_2)) \rightarrow (u_0(x_2), 0) \quad \text{as } x_1 \rightarrow -\infty. \quad (1.5)$$

Finally, the density in the upstream is given as follows

$$\rho(x_1, x_2) \rightarrow \rho_0 \quad \text{as } x_1 \rightarrow -\infty \quad (1.6)$$

where  $\rho_0$  is a constant. Furthermore, if  $f$  is not differentiable at  $x_1 = 0$  and  $1$ , then the flow is required to satisfy the Kutta-Joukowski condition (cf. [1]) at the corner points  $\{P_1, P_2\}$  where  $P_1 = (0, 0)$  and  $P_2 = (1, 0)$ , i.e., the flow velocity is continuous at  $\{P_1, P_2\}$ .

Let us state the main results as follows.

**Theorem 1.1.** *Suppose that the upstream horizontal velocity  $u_0(x_2)$  in (1.5) satisfies that*

$$u_0(x_2) \in C^2(\mathbb{R}^+), \quad u_0(x_2) > 0, \quad u_0''(x_2) \geq 0, \quad u_0'(0) \leq 0, \quad \lim_{x_2 \rightarrow +\infty} u_0'(x_2) = 0, \quad (1.7)$$

*and there exists a  $\bar{u} > 0$  such that*

$$\lim_{x_2 \rightarrow +\infty} u_0(x_2) = \bar{u}, \quad (1.8)$$

*then there exists a critical value  $\rho_{cr} > 0$ , such that if the incoming density  $\rho_0$  in (1.6) is larger than  $\rho_{cr}$ , then there exists a uniformly subsonic flow  $(\rho, u, v) \in \left(C^{1,\alpha}(\Omega) \cap C^\beta(\bar{\Omega})\right)^3$  for some  $\beta \in (0, \alpha)$ , which satisfies the Euler system (1.1), the boundary conditions (1.4)-(1.6), and Kutta-Joukowski condition at the corner points  $\{P_1, P_2\}$ . Moreover,*

(1) *the flow is uniformly subsonic*

$$\sup_{\bar{\Omega}} (u^2 + v^2 - c^2(\rho)) < 0, \quad (1.9)$$

*and the horizontal velocity is positive except at the corners*

$$u > 0 \quad \text{in } \bar{\Omega} \setminus \{P_1 \cup P_2\} \quad \text{and } u = 0 \quad \text{at } P_1 \cup P_2; \quad (1.10)$$

(2) the flow satisfies

$$\|(\rho u - \rho_0 u_0(x_2), \rho v)\|_{L^2(\Omega)} \leq C \quad (1.11)$$

for some  $C > 0$  and has the following asymptotic behavior in far fields,

$$\rho \rightarrow \rho_0, \quad (u, v) \rightarrow (u_0(x_2), 0), \quad (1.12)$$

$$\nabla \rho \rightarrow 0, \quad \nabla u \rightarrow (0, u'_0(x_2)), \quad \nabla v \rightarrow 0, \quad (1.13)$$

as  $|x_1| \rightarrow \infty$  uniformly for  $x_2 \in K_1 \Subset (0, +\infty)$ , and

$$u \rightarrow \bar{u}, \quad v \rightarrow 0, \quad \rho \rightarrow \rho_0, \quad \text{as } x_2 \rightarrow +\infty; \quad (1.14)$$

(3) the subsonic flow satisfying Euler system (1.1), boundary conditions (1.4)-(1.6), (1.10), and asymptotic behavior (1.11)-(1.13) is unique.

(4)  $\rho_{cr}$  is the critical incoming density for the existence of subsonic flow past a wall in the following sense: either

$$\sup_{\Omega} \frac{|(u, v)|}{c(\rho)} \rightarrow 1 \quad \text{as } \rho_0 \rightarrow \rho_{cr}, \quad (1.15)$$

or there is no  $\sigma > 0$ , such that for all  $\rho_0 \in (\rho_{cr} - \sigma, \rho_{cr})$  there are Euler flows satisfying (1.1), subsonic condition (1.9) and asymptotic behavior (1.11)-(1.13), and

$$\sup_{\rho_0 \in (\rho_{cr} - \sigma, \rho_{cr})} \sup_{\Omega} \frac{|(u, v)|}{c(\rho)} < 1.$$

If the convexity assumption on  $u_0$  in (1.7) is removed, then the following theorem holds.

**Theorem 1.2.** Suppose that the horizontal velocity  $u_0(x_2)$  in the upstream satisfies that

$$u_0(x_2) \in C^2(\mathbb{R}^+), \quad u_0(x_2) > 0, \quad u'_0(0) \leq 0, \quad \lim_{x_2 \rightarrow \infty} u_0(x_2) = \bar{u}, \quad (1.16)$$

for some constant  $\bar{u} > 0$ . Furthermore, there exist constants  $k > 1$  and  $\varepsilon > 0$  such that

$$\left| \frac{d^i u_0(x_2)}{dx_2^i} \right| \leq \frac{\varepsilon}{(1 + x_2)^{k+i}} \quad \text{for } i = 1, 2 \quad \text{and } x_2 > 0. \quad (1.17)$$

There exist an  $\varepsilon_0 > 0$  and a critical value  $\rho_{cr} > 0$  such that if  $\varepsilon$  in (1.17) satisfies  $\varepsilon \in (0, \varepsilon_0)$ , and the incoming density  $\rho_0$  in the upstream is larger than  $\rho_{cr}$ , then the problem (1.1), (1.4)-(1.6) admits a uniformly subsonic flow  $(\rho, u, v) \in \left( C^{1,\alpha}(\Omega) \cap C^\beta(\bar{\Omega}) \right)^3$  for some  $\beta \in (0, \alpha)$  satisfying Kutta-Joukowski condition at the corner points  $\{P_1, P_2\}$ . Moreover, the properties (1)-(4) in Theorem 1.1 hold.

As a direct consequence of Theorem 1.2, one gets the existence and uniqueness of smooth subsonic ideal flows past a symmetric obstacle.

**Corollary 1.3.** (Subsonic flow past a symmetric obstacle) Let  $\tilde{\Omega}$  be defined as

$$\tilde{\Omega} = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_2| > f(x_1)\}.$$

Suppose that the horizontal velocity  $u_0(x_2)$  in the upstream satisfies

$$u_0(x_2) \in C^2(\mathbb{R}), \quad u_0(x_2) = u_0(-x_2), \quad u_0(x_2) > 0, \quad \lim_{x_2 \rightarrow \pm\infty} u_0(x_2) = \bar{u} > 0, \quad (1.18)$$

and (1.17), then there exists a critical value  $\rho_{cr} > 0$ , such that if the incoming density  $\rho_0$  in the upstream is larger than  $\rho_{cr}$ , then there exists a uniformly subsonic symmetric flow  $(u, v, \rho) \in \left(C^{1,\alpha}(\tilde{\Omega}) \cap C^\beta(\bar{\tilde{\Omega}})\right)^3$ , which satisfies the Euler system (1.1), the boundary conditions (1.4)-(1.6). Moreover, the similar properties for  $(\rho, u, v)$  as that in (1)-(4) in Theorem 1.1 hold.

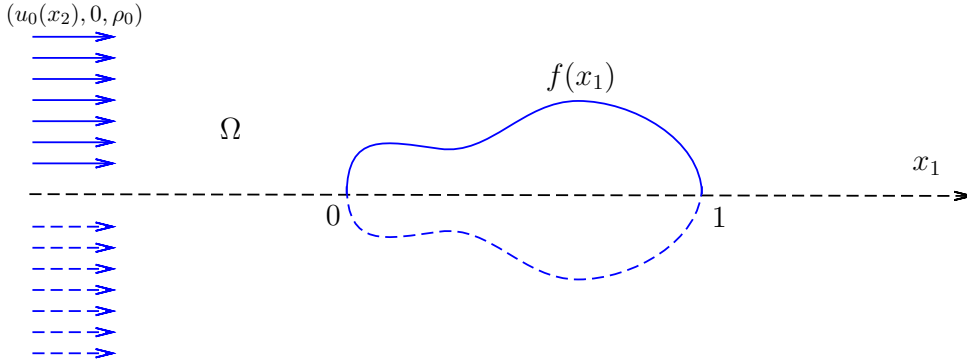


FIGURE 2. Subsonic flows past a symmetric obstacle

As  $\rho_0$  approaches to  $\rho_{cr}$  in Theorems 1.1 and 1.2 and Corollary 1.3, combining with the compensated compactness framework obtained in [8], we have the following theorem.

**Theorem 1.4.** Assume that  $\{(\rho_n, u_n, v_n)\}$  is a sequence of subsonic Euler flows in  $\Omega$  or  $\tilde{\Omega}$  which satisfy (1.4)-(1.6) with  $\rho_0$  replaced by  $\rho_0^{(n)}$ . Suppose that  $\rho_0^{(n)} \downarrow \rho_{cr}$  as  $n \rightarrow \infty$ , then there exists a limiting flow  $(\rho, u, v)$  satisfying

$$(\rho_n, u_n, v_n) \rightarrow (\rho, u, v) \text{ a.e. in } \Omega \text{ as } n \rightarrow +\infty. \quad (1.19)$$

Furthermore,  $(\rho, u, v)$  satisfies the Euler system (1.1) in the sense of distribution and the boundary condition (1.4) in the sense of normal trace.

There are several remarks in order for the main results.

**Remark 1.1.** The most significant difference between the Euler flows with non-zero vorticity and irrotational flows is that the governing system for subsonic flows with non-zero vorticity is a hyperbolic-elliptic coupled system. In order to solve the hyperbolic mode, one has to prescribe suitable physical boundary conditions for the corresponding hyperbolic mode. The far field boundary condition (1.6) can be regarded as the boundary condition for the associated hyperbolic mode. The general form for the boundary conditions in the upstream can be written as

$$(\rho(x_1, x_2), u(x_1, x_2), v(x_1, x_2)) \rightarrow (\rho_0(x_2), u_0(x_2), v_0(x_2)) \quad \text{as } x_1 \rightarrow -\infty. \quad (1.20)$$

Suppose that the flows satisfy the far field conditions with high order compatibility conditions, i.e.,

$$\nabla(\rho(x_1, x_2), u(x_1, x_2), v(x_1, x_2)) \rightarrow \nabla(\rho_0(x_2), u_0(x_2), v_0(x_2)) \quad \text{as } x_1 \rightarrow -\infty. \quad (1.21)$$

It follows from the Euler system (1.1) that one has

$$(\rho_0 v_0)_{x_2} = 0, \quad (\rho_0 u_0 v_0)_{x_2} = 0, \quad (\rho_0 v_0^2 + p(\rho_0))_{x_2} = 0. \quad (1.22)$$

The first equation in (1.22) shows that

$$\rho_0 v_0 \equiv C_1 \quad (1.23)$$

for some constant  $C_1$ . The slip boundary condition (1.4) yields that  $\rho_0(0)v_0(0) = 0$ . Hence  $C_1 = 0$ . Since we are looking for steady subsonic solutions,  $\rho_0 \neq 0$ . Thus  $v_0 \equiv 0$ . It follows from the third equation in (1.22) that  $(p(\rho_0))_{x_2} = 0$ . This yields that  $\rho_0$  must be a constant. Therefore, the boundary conditions (1.5) and (1.6) are indeed the general form for the boundary conditions for subsonic flows past a wall or symmetric body if the solutions satisfy have high order consistency with the boundary conditions.

**Remark 1.2.** For subsonic flows with non-zero vorticity, the boundary condition (1.6) is used to solve the hyperbolic mode in the steady Euler system. When the far field velocity in (1.5) is a constant field, the flows obtained in this paper are subsonic or subsonic-sonic irrotational flows. In fact, the boundary condition (1.6) together with (1.5) can be used to determine Bernoulli constant for subsonic irrotational flows past a body when the far field velocity in (1.5) is a constant. In this case, the boundary conditions (1.4)-(1.6) and Kutta-Joukowski condition at the corner points  $\{P_1, P_2\}$  reduces to the boundary conditions for subsonic irrotational flows past a body [2]. Therefore, the boundary conditions we prescribed for subsonic Euler flows past a wall not only are suitable physical boundary conditions for the well-posedness theory for subsonic flows with non-zero vorticity past a wall, but also can be regarded as generalizations for the boundary conditions for irrotational flows past a wall,

**Remark 1.3.** When the streamlines of the flows have simple topological structure, we use the stream function formulation to solve the hyperbolic mode in the Euler system so that the hyperbolic-elliptic coupled Euler system is reduced to a single second order quasilinear equation with memory. Therefore, our first key task is to show that the flows past a wall indeed have simple topological structure. Second, even for the problem for the stream function, the memory term makes the equation more complicated than the one for irrotational flows. The governing equation for irrotational flows is a homogenous second order quasilinear equation for which the maximum principle can be applied easily [19]. Furthermore, the far field of the irrotational flows past a body is a uniform constant state so that the far fields can be essentially regarded as a single point and the problem can be transformed into a bounded domain problem [2, 19, 20, 11]. The memory term for the flows with non-zero vorticity yields non-uniform far field states. Therefore, the far fields for the flows with non-zero vorticity

cannot be regarded as a single point so that the problem for subsonic flows with non-zero vorticity past a wall is indeed a problem on a unbounded domain.

**Remark 1.4.** We use a sequence of nozzles to approximate the region above the wall so that the ideas in [40, 15, 16] can be used to get approximated solutions. However, the estimates developed in [40, 15, 16] depend on the height of the nozzles. In order to get the existence for flows past a wall, we have to establish some uniform estimates for the flows in the approximated nozzles independent of nozzle height. This is also the crucial step to obtain the existence of subsonic Euler flows past a wall.

**Remark 1.5.** The uniqueness of solution in Theorems 1.1 and 1.2 and Corollary 1.3 is obtained in the class of subsonic Euler flows in which we proved the existence, i.e. the class of flows which also have positive horizontal velocity and satisfy the asymptotic behavior (1.11)-(1.14). In the future, we hope to prove the uniqueness of subsonic Euler flows which satisfy only the the boundary conditions (1.4)-(1.6) and Kutta-Joukowski condition as in the case of irrotational flows in [19].

**Remark 1.6.** The flows obtained in Theorem 1.1 may have large vorticity so that they may not be close to potential flows, which cannot be treated as small perturbations of potential flows and the effect of vorticity in the whole domain has to be analyzed.

**Remark 1.7.** Since the flows have stagnation points at those corner points, it is not easy to claim the absence of stagnation point inside the domain. In order to solve the hyperbolic mode in the Euler system, we need to show the absence of stagnation points inside the flow region and get precise far field behavior of the flows so that the hyperbolic-elliptic coupled Euler system can be reduced to a second order quasilinear equation with memory term for the stream function. Although the flows obtained in Theorem 1.2 have small vorticity, the memory term appeared in the equation for the stream function makes it hard to prove far field behavior of the flows and the absence of stagnation point above the wall.

**Remark 1.8.** As mentioned in [9, Remark 6.1], the absence of stagnation point in the flow region was not obtained in [9] so that the equivalence between the stream function formulation and the Euler system, in particular, the existence and uniqueness of the solutions of the original Euler system are not established in [9]; we prove the existence and uniqueness of subsonic flows for the original Euler system when the incoming density is greater than a critical value as long as the horizontal velocity is positive convex or a small perturbation of a constant state, and we also show that the flows do not have stagnation point above the wall. Even for the problem for the stream function (cf. the problem (2.17)), the existence and uniqueness of the solutions were obtained in [9] only when the incoming velocity is a sufficiently smaller perturbation of a small constant state and the incoming density is a large constant; while we can deal with the flows with large vorticity and the density which is greater than a critical value. Finally, the crucial estimates in [9] are for elliptic equations which are small perturbations of two dimensional Laplace equation so that the author in [9]



can only deal with the flows with small velocity and vorticity; in order to deal with subsonic flows whose Mach number might approach to one, we have to rebuild the estimate for general quasilinear equation for the stream functions.

**Remark 1.9.** Note that we only obtained the weak convergence in Theorem 1.4. We hope to study the regularity and fine structure of these limiting solutions later on. Continuous transonic irrotational flows in nozzles were constructed in [33, 34, 35] recently. Furthermore, there are a lot of studies on stability of transonic shock solutions in nozzles recently, see [7, 10, 25, 26, 41, 42] and references therein. For the problems on the stability of transonic shocks, different boundary conditions for the flows at the downstream are prescribed, where the key issue is to study the free boundary problem for subsonic flows which are usually small perturbations of some given background solutions.

**Remark 1.10.** Theorems 1.1 and 1.2 can be extended easily to subsonic Euler flows past several smooth bumps (See Figure 3).

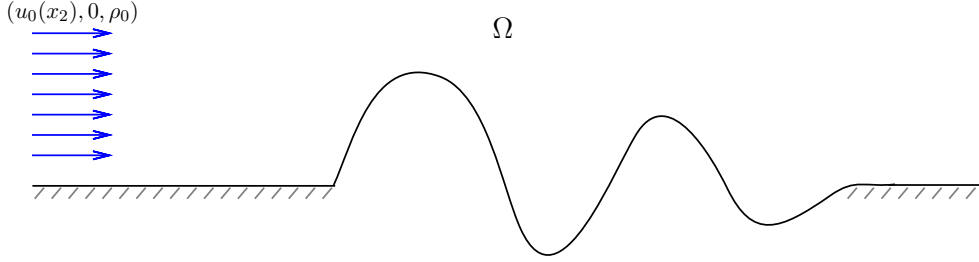


FIGURE 3. Subsonic flows past several bumps

**Remark 1.11.** Corollary 1.3 also holds for the subsonic flows past a symmetric body with a cusp (See Figure 4). The only difference is the horizontal velocity is not necessary zero at the trailing edge.

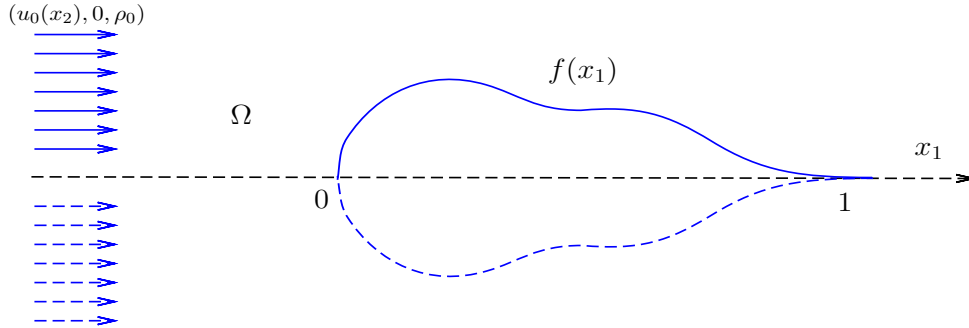


FIGURE 4. Subsonic flows past a symmetric obstacle with cusp

**Remark 1.12.** All results in this paper also hold for the non-isentropic Euler equations, provided that we impose the entropy of the flows in the upstream.

Before giving the detailed proof for these main results, we present the main ideas of the proof as follows and the recent progress on some problems related to flows past a body. Adapting the stream function formulation developed in [40], we reduce the Euler system into a single second order elliptic equation with memory for the stream function. In order to deal with the associated problem for the stream function in the region above the wall, we approximate the domain by a sequence of infinitely long nozzles  $\Omega_L$  bounded by  $\Gamma$  and  $\Gamma_L = \{(x_1, x_2) | x_2 = L\}$ . Combining the ideas and techniques in [40, 15, 16] gives the existence of subsonic Euler flows  $(\rho_L, u_L, v_L)$  in  $\Omega_L$ . One of the key issues in this paper is to obtain some uniform estimates for the solutions  $(\rho_L, u_L, v_L)$  independent of  $L$ . The uniform estimates consist of mainly two parts. The first one concerns the  $L^\infty$  estimate for the difference of the stream function and the corresponding one in the upstream, which is based on a maximum principle. The second one is the uniform  $L^2$  gradient estimate for the same quantity, which follows from a delicate energy estimate and is used to prove the asymptotic behavior and the uniqueness of the flows.

The key part of the proof is on the uniform estimate for the steady Euler flows in infinitely long nozzles. In fact, the steady Euler flows in nozzles were studied extensively recently. In his famous survey [1], Bers asserted that there exists a unique subsonic irrotational flow in a two-dimensional infinitely long nozzle, provided the incoming mass flux is sufficiently small. The rigorous mathematical proof to this assertion was obtained in [38] for flows in 2D nozzles, in [39] for flows in axially symmetric nozzles, and [17] for flows in arbitrary multi-dimensional nozzles, respectively. Furthermore, the existence of subsonic irrotational flows was established in [17, 38, 39] as long as the mass flux is less than a critical value. When the mass flux approaches the critical value, the associated flows converge weakly to subsonic-sonic flows, see [24, 38, 39]. As we mentioned before, subsonic flows with non-zero vorticity were first studied in [40] via a stream function formulation, which was generalized in [3, 12, 13, 6, 18, 15, 16] and reference therein. Subsonic Euler flows in finite nozzles, in particular, three dimensional nozzles were investigated recently in [4, 14, 37, 36].

The rest of this paper is organized as follows. We first adapt the stream function formulation for the steady Euler flows developed in [40] to reduce the steady Euler system into a single second order equation in Section 2. To obtain the existence of the subsonic solution to the equation for stream function above a wall, we construct a series of uniformly subsonic solutions in infinitely long nozzles as the approximated solutions. Some uniform estimates for the approximated solutions are obtained in Section 3 so that the existence of subsonic solution above a wall is established when the density of the incoming flows is sufficiently large. In order to show that the solution induced by the stream function solves the compressible Euler system, in Section 4 we prove the asymptotic behavior of the flows in the far fields and show that the horizontal velocity of the flows is always positive except at the corners on the

boundary. The uniqueness of a subsonic flow past a wall is given in Section 5 by the energy method with the aid of  $L^2$  integral estimates. In Section 6, the existence of the critical value of the density in the upstream is established so that the proof of Theorem 1.1 is completed. The Section 7 sketches the proof for the existence and the uniqueness of the compressible subsonic Euler flow past a wall when the upstream horizontal velocity is a general small perturbation of a constant, which gives the proof of Theorem 1.2. Theorem 1.4 is proved in Section 8 with the help of the compensated compactness framework established in [8]. A weighted Poincaré inequality used in Section 7 and its proof are presented in Appendix A.

## 2. REFORMULATION OF THE PROBLEM

It follows from the Euler system (1.1) that one has

$$(u, v) \cdot \nabla \left( \frac{\omega}{\rho} \right) = 0 \quad (2.1)$$

and

$$(u, v) \cdot \nabla B = 0, \quad (2.2)$$

where

$$\omega = \partial_{x_1} v - \partial_{x_2} u \quad \text{and} \quad B = \frac{u^2 + v^2}{2} + \frac{\gamma \rho^{\gamma-1}}{\gamma - 1}$$

are the vorticity and the Bernoulli function of the flows, respectively. The equation (2.2) is nothing but the Bernoulli's law. Later on, we denote  $h(\rho) = \frac{\gamma \rho^{\gamma-1}}{\gamma - 1}$  which is called the enthalpy of the polytropic gas.

**Proposition 2.1.** *Suppose that*

$$\rho > 0 \text{ and } u > 0 \text{ in } \Omega, \text{ and } u \geq \delta \text{ for } |x| \geq R \quad (2.3)$$

*for some positive constants  $\delta$  and  $R$ . Furthermore, assume that*

$$u, \rho, \text{ and } v_{x_2} \text{ are bounded, while } v, v_{x_1}, \text{ and } \rho_{x_2} \rightarrow 0, \text{ as } x_1 \rightarrow \pm\infty. \quad (2.4)$$

*For a smooth flow in  $\Omega$  satisfying (2.3) and (2.4), the Euler system (1.2) is equivalent to the continuity equation, (2.1) and (2.2).*

*Proof.* A similar result for flows in infinitely long nozzles has been proved in [40, Proposition 1]. We sketch the proof for the flows past a wall as follows.

Note that the equations (2.1) and (2.2) can be concluded from the Euler system (1.1). Hence, it suffices to prove the validity of the Euler system (1.1) from the continuity equation and (2.1)-(2.2). It follows from the continuity equation and (2.1) that

$$\partial_{x_2}(uu_{x_1} + vu_{x_2} + p_{x_1}/\rho) - \partial_{x_1}(uv_{x_1} + vv_{x_2} + p_{x_2}/\rho) = 0.$$

Therefore, there exists a function  $\Phi$  such that

$$\partial_{x_1}\Phi = uu_{x_1} + vu_{x_2} + p_{x_1}/\rho \text{ and } \partial_{x_2}\Phi = uv_{x_1} + vv_{x_2} + p_{x_2}/\rho. \quad (2.5)$$

Furthermore, the straightforward computations show that

$$(u, v) \cdot \nabla \Phi = (u, v) \cdot \nabla B. \quad (2.6)$$

Thus  $(u, v) \cdot \nabla \Phi = 0$ . Since the horizontal velocity is positive above the wall, each streamline defined by the equation

$$\begin{cases} \frac{dx_1}{ds} = u(x_1(s), x_2(s)), \\ \frac{dx_2}{ds} = v(x_1(s), x_2(s)) \end{cases} \quad (2.7)$$

can be stretched to a unique position in the upstream. To see this, we need only to claim that any streamline through a point in  $\Omega$  cannot touch the wall or go to the infinity in the  $x_2$  direction. In view of the argument in [40, Proposition 1], such a streamline is away from the wall. Thus, we need only to show that the streamline cannot go to infinity in the  $x_2$ -direction with finite  $x_1$ . In fact, the assumption (2.3) that  $u$  is uniformly positive for large  $x_2$  implies that  $\left|\frac{v}{u}\right|$  is uniformly bounded. So, the solution of the ODE

$$\frac{dx_2}{dx_1} = \frac{v}{u}(x_1, x_2)$$

cannot blow up at finite  $x_1$ . It follows from (2.4) that  $\Phi$  is a constant as  $x_1 \rightarrow -\infty$ . Since  $\Phi$  is a constant along each streamline,  $\Phi$  is a constant in  $\Omega$ . Using (2.5) yields

$$uu_{x_1} + vu_{x_2} + p_{x_1}/\rho = 0 \quad \text{and} \quad uv_{x_1} + vv_{x_2} + p_{x_2}/\rho = 0.$$

These, together with the continuity equation give the Euler system (1.2).  $\square$

It follows from the continuity equation that there exists a stream function  $\psi$  satisfying

$$\psi_{x_2} = \rho u \quad \text{and} \quad \psi_{x_1} = -\rho v. \quad (2.8)$$

This, together with the slip boundary condition (1.4), shows that  $\psi$  is a constant on the boundary  $\Gamma$ . Without loss of generality, we assume that  $\psi = 0$  on the boundary  $\Gamma$ .

If the flow satisfies (2.3), then the streamlines possess a simple topological structure in the whole domain, so that we can parameterize the streamlines in the domain by their positions in the upstream. By the definition of the stream function, we have the following parametrization for the stream function in the upstream (See Figure 5)

$$\psi = \int_0^{\kappa(\psi; \rho_0)} \rho_0 u_0(s) ds. \quad (2.9)$$

Denote

$$F(\psi; \rho_0) = u_0(\kappa(\psi; \rho_0)). \quad (2.10)$$

The Bernoulli's law yields

$$\frac{|\nabla \psi|^2}{2\rho^2} + h(\rho) = h(\rho_0) + \frac{1}{2}F^2(\psi; \rho_0) = \mathcal{B}(\psi; \rho_0). \quad (2.11)$$

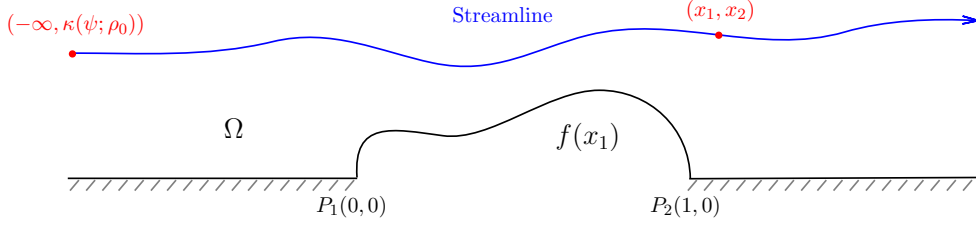


FIGURE 5. Parameterize the streamlines

For given  $\mathfrak{s}$ , there exist unique  $\underline{\varrho}(\mathfrak{s})$  and  $\bar{\varrho}(\mathfrak{s})$  such that

$$\frac{1}{2}\gamma\underline{\varrho}^{\gamma-1}(\mathfrak{s}) + h(\underline{\varrho}(\mathfrak{s})) = \mathfrak{s} \quad \text{and} \quad h(\bar{\varrho}(\mathfrak{s})) = \mathfrak{s}. \quad (2.12)$$

Define

$$\Sigma(\mathfrak{s}) = \sqrt{\gamma}\underline{\varrho}^{\frac{\gamma+1}{2}}(\mathfrak{s}) \quad (2.13)$$

which is the critical momentum associated with the sonic state. It is easy to check from

$$\frac{\mathfrak{m}^2}{2\rho^2} + h(\rho) = \mathfrak{s} \quad (2.14)$$

that for given  $\mathfrak{s}$ , the momentum  $\mathfrak{m}$  is a strictly decreasing function of  $\rho$  for  $\rho \in (\underline{\varrho}(\mathfrak{s}), \bar{\varrho}(\mathfrak{s}))$ . Therefore, for given  $\mathfrak{s}$  and  $\mathfrak{m}$ , one can find a unique  $\rho \in (\underline{\varrho}(\mathfrak{s}), \bar{\varrho}(\mathfrak{s}))$  satisfying (2.14) and denote it by

$$\rho = \mathcal{H}(\mathfrak{m}^2, \mathfrak{s}). \quad (2.15)$$

Later on, we also write

$$\rho = \mathcal{H}(|\nabla\psi|^2, \mathcal{B}(\psi; \rho_0)) = H(|\nabla\psi|^2, \psi; \rho_0)$$

to emphasize the dependence on the parameter  $\rho_0$ .

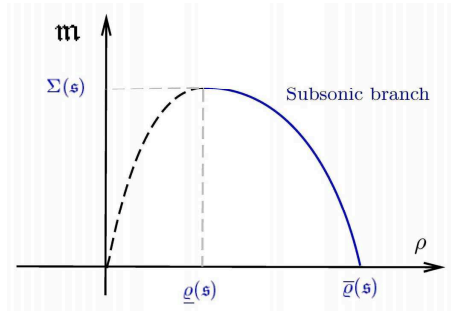


FIGURE 6. The relationship between the momentum and the density

It follows from the vorticity equation (2.1) that the scaled vorticity  $\frac{\omega}{\rho}$  is conserved along each streamline. Thus, it holds that

$$\omega = -\frac{H(|\nabla\psi|^2, \psi; \rho_0) u'_0(\kappa(\psi; \rho_0))}{\rho_0}.$$

This, together with the identity  $\omega = -\operatorname{div} \left( \frac{\nabla \psi}{\rho} \right)$ , gives the equation for the stream function  $\psi$  as follows

$$\operatorname{div} \left( \frac{\nabla \psi}{H(|\nabla \psi|^2, \psi; \rho_0)} \right) = u'_0(\kappa(\psi; \rho_0)) H(|\nabla \psi|^2, \psi; \rho_0) / \rho_0. \quad (2.16)$$

The straightforward computations for (2.9) and (2.10) yield

$$u'_0(\kappa(\psi; \rho_0)) = \rho_0 F(\psi; \rho_0) F'(\psi; \rho_0),$$

where  $F'$  is the derivative with respect to  $\psi$  of  $F$ . Therefore, the problem for the Euler system (1.1) with boundary condition (1.4) is equivalent to

$$\begin{cases} \operatorname{div} \left( \frac{\nabla \psi}{H(|\nabla \psi|^2, \psi; \rho_0)} \right) = F(\psi; \rho_0) F'(\psi; \rho_0) H(|\nabla \psi|^2, \psi; \rho_0) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.17)$$

Note that the equation in (2.17) can be written as

$$\left( \left( 1 - \frac{|\nabla \psi|^2}{\gamma H^{\gamma+1}} \right) \delta_{ij} + \frac{\partial_i \psi \partial_j \psi}{\gamma H^{\gamma+1}} \right) \partial_{ij} \psi = F(\psi; \rho_0) F'(\psi; \rho_0) H^2. \quad (2.18)$$

It is easy to see that the coefficient matrix of the equation (2.18) has two eigenvalues  $\lambda = 1 - \frac{|\nabla \psi|^2}{\gamma H^{\gamma+1}}$  and  $\Lambda = 1$ . As  $|\nabla \psi|^2 \rightarrow \gamma H^{\gamma+1}$ , i.e., the flows go to sonic, the equation (2.18) becomes degenerate elliptic. This is one of the major difficulty for solving the problem (2.17). Another main difficulty here is that the domain  $\Omega$  is unbounded in every direction.

### 3. EXISTENCE OF SUBSONIC SOLUTION WITH LARGE INCOMING DENSITY

In this section, we establish the existence of subsonic solutions for the boundary value problem (2.17) when the incoming density  $\rho_0$  is sufficiently large. Since the upstream is uniformly subsonic, the uniform density  $\rho_0$  in the upstream indeed has a lower-bound  $\rho_0^* = \left( \frac{1}{\gamma} \sup u_0^2(x_2) \right)^{\frac{1}{\gamma}}$ . In Sections 3-6, we assume that the hypotheses in Theorem 1.1 hold. First, we have the following proposition.

**Proposition 3.1.** *There exists a  $\bar{\rho}_0 \in (\rho_0^*, \infty)$  such that if  $\rho_0 > \bar{\rho}_0$ , the problem (2.17) has a subsonic solution satisfying*

$$0 \leq \psi \leq \bar{\psi}, \quad \bar{\psi} - \psi \leq C \rho_0, \quad \text{and} \quad \sup_{x \in \Omega} \frac{|\nabla \psi|}{\gamma H^{\frac{\gamma+1}{2}}(|\nabla \psi|^2, \psi; \rho_0)} \leq \frac{1}{4}, \quad (3.1)$$

where

$$\bar{\psi} = \rho_0 \int_0^{x_2} u_0(s) ds \quad (3.2)$$

and the constant  $C$  depends only on  $\max f(x_1)$  and  $\max u_0(x_2)$ .

The problem (2.17) for subsonic flows is a Dirichlet problem for a quasilinear elliptic equation. The domain  $\Omega$  is unbounded in both  $x_1$  and  $x_2$  directions so that the stream function becomes an unbounded function, which is one of the main differences from the problem of subsonic flows in infinitely long nozzles. To overcome this difficulty, we first establish the approximated problems in some infinitely long nozzles and then obtain the existence of the subsonic solution in  $\Omega$  by the uniform estimates for the approximated solutions.

**3.1. Existence of subsonic solutions in nozzles.** Let  $J = \sup f(x_1)$ . Given  $L \in \mathbb{N}$  satisfying  $L > J$ , define

$$\Omega_L = \{x \in \Omega \mid f(x_1) < x_2 < L\} \quad \text{and} \quad \Gamma_L = \{(x_1, L) \mid x_1 \in \mathbb{R}\}. \quad (3.3)$$

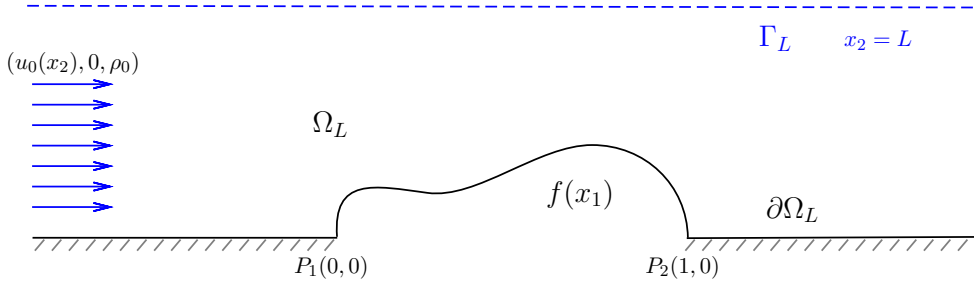


FIGURE 7. Truncated domain  $\Omega_L$

Let

$$g_L(x_2) = \begin{cases} u'_0(x_2) & \text{if } x_2 \leq L-1, \\ u'_0(L-1)(L-x_2) & \text{if } L-1 < x_2 \leq L, \end{cases} \quad (3.4)$$

and  $u_{0,L}(x_2) = u_0(0) + \int_0^{x_2} g_L(s)ds$ . Hence, it follows that for  $x_2 \in [0, L]$ ,

$$u_{0,L}(x_2) \geq u_0(L-1) + \frac{u'_0(L-1)}{2}.$$

If  $L$  is sufficiently large, then  $u_{0,L}(x_2) \geq \bar{u}/2$  for all  $x_2 \in [0, L]$ . Furthermore,  $u_{0,L}(x_2)$  satisfies  $u'_{0,L}(L) = g_L(L) = 0$  and

$$u''_{0,L}(x_2) = g'_L(x_2) = \begin{cases} u''_0(x_2) & \text{if } x_2 \leq L-1, \\ -u'_0(L-1) & \text{if } L-1 < x_2 \leq L. \end{cases} \quad (3.5)$$

Therefore,  $u''_{0,L}(x_2) \geq 0$  for  $x_2 \in [0, L]$ .

Let  $\kappa_L(\psi; \rho_0)$  satisfy

$$\psi = \int_0^{\kappa_L(\psi; \rho_0)} \rho_0 u_{0,L}(s) ds. \quad (3.6)$$

Denote

$$F_L(\psi; \rho_0) = u_{0,L}(\kappa_L(\psi; \rho_0)), \quad W_L(\psi) = F_L(\psi; \rho_0)F'_L(\psi; \rho_0), \quad m_L =: \int_0^L \rho_0 u_{0,L}(s) ds, \quad (3.7)$$

and

$$\mathcal{B}_L(\psi; \rho_0) = h(\rho_0) + \frac{1}{2}F_L^2(\psi; \rho_0) \quad \text{and} \quad H_L(|\nabla\psi|^2, \psi; \rho_0) = \mathcal{H}(|\nabla\psi|^2, \mathcal{B}_L(\psi; \rho_0)). \quad (3.8)$$

We consider the problem

$$\begin{cases} \left( \left( 1 - \frac{|\nabla\psi|^2}{\gamma H_L^{\gamma+1}} \right) \delta_{ij} + \frac{\partial_i \psi \partial_j \psi}{\gamma H_L^{\gamma+1}} \right) \partial_{ij} \psi = W_L H_L^2 & \text{in } \Omega_L, \\ \psi = 0 & \text{on } \Gamma, \quad \psi = m_L & \text{on } \Gamma_L. \end{cases} \quad (3.9)$$

The problem (3.9) has been studied in detail in [40, 16, 15]. For the convenience, we give a sketch for the study of (3.9) in this subsection. There are several difficulties for this problem. First,  $\mathcal{B}_L$  is not well-defined if  $\psi \notin [0, m_L]$ . Second, the equation in (3.9) is degenerate as  $\frac{|\nabla\psi|^2}{\gamma H_L^{\gamma+1}}$  approaches to 1. Finally, the problem is on an unbounded domain.

It is easy to see that

$$F'_L(0; \rho_0) \leq 0 \quad \text{and} \quad F'_L(m_L; \rho_0) = 0. \quad (3.10)$$

If we extend  $F_L$  to  $\tilde{F}_L$  such that

$$\tilde{F}_L(s; \rho_0) = \begin{cases} F_L(s; \rho_0) & \text{if } s \in [0, m_L], \\ F_L(m_L; \rho_0) & \text{if } s > m_L, \\ F_L(0; \rho_0) + \frac{F'_L(0; \rho_0)}{2} \left( s + \frac{s^2}{2} \right) & \text{if } -1 \leq s < 0, \\ F_L(0; \rho_0) - \frac{F'_L(0; \rho_0)}{2} & \text{if } s < -1, \end{cases} \quad (3.11)$$

then  $\tilde{F}_L$  is a continuous function. Define

$$\tilde{W}_L(\psi; \rho_0) = \tilde{F}_L(\psi; \rho_0) \tilde{F}'_L(\psi; \rho_0) \quad \text{and} \quad \tilde{\mathcal{B}}_L(\psi; \rho_0) = h(\rho_0) + \frac{1}{2} \tilde{F}_L^2(\psi; \rho_0). \quad (3.12)$$

In order to deal with the possible degeneracy appeared in the equation in (3.9), as in [40, 16, 15], we first truncate the associated equation. Let  $\chi(s)$  be a smooth increasing odd function satisfying

$$\zeta(s) = \begin{cases} s & \text{if } |s| \leq 1/2, \\ 5/8 & \text{if } s > 3/4. \end{cases} \quad (3.13)$$



Define  $\check{H}_L(|\nabla\psi|^2, \psi; \rho_0) = \mathcal{H}\left(\zeta^2\left(\frac{|\nabla\psi|}{\Sigma(\check{\mathcal{B}}_L(\psi; \rho_0))}\right)\Sigma^2(\check{\mathcal{B}}_L(\psi; \rho_0)), \check{\mathcal{B}}_L(\psi; \rho_0)\right)$  where  $\Sigma$  is the function defined in (2.13). Instead of (3.9), we first study the following problem

$$\begin{cases} \left(\left(1 - \zeta^2\left(\frac{|\nabla\psi|}{\sqrt{\gamma}\check{H}_L^{\frac{\gamma+1}{2}}}\right)\right)\delta_{ij} + \zeta\left(\frac{\partial_i\psi}{\sqrt{\gamma}\check{H}_L^{\frac{\gamma+1}{2}}}\right)\zeta\left(\frac{\partial_j\psi}{\sqrt{\gamma}\check{H}_L^{\frac{\gamma+1}{2}}}\right)\right)\partial_{ij}\psi = \check{W}_L\check{H}_L^2 & \text{in } \Omega_L, \\ \psi = 0 & \text{on } \Gamma, \quad \psi = m_L & \text{on } \Gamma_L. \end{cases} \quad (3.14)$$

Later on, a constant is said to depend on the elliptic coefficients, if it depends on the coefficients of an elliptic equation.  $C$  denotes a constant independent of  $L$  and the elliptic coefficients,  $\mathcal{C}$  denotes a constant which is independent of  $L$  but depends on the elliptic coefficients, and  $\mathfrak{C}$  is a constant depending both  $L$  and elliptic coefficients. The value of these constants may vary from line to line but they keep the same property.

**Lemma 3.2.** *For any  $\rho_0 > \rho_0^*$ , there exists a solution  $\psi_L \in C^{2,\alpha}(\Omega_L) \cap C^{1,\beta}(\bar{\Omega}_L)$  to the problem (3.14) satisfying*

$$0 \leq \psi_L \leq m_L \quad \text{in } \Omega_L. \quad (3.15)$$

Furthermore, there exists a  $\check{\rho}_{0,L} \in (\rho_0^*, \infty)$  such that if  $\rho_0 > \check{\rho}_{0,L}$ , it holds that

$$\max \frac{|\nabla\psi(x; \rho_0)|}{\Sigma(\mathcal{B}_L(\psi; \rho_0))} < \frac{1}{4}. \quad (3.16)$$

The proof of this lemma is a combination of [15, Section 2.2] and [16, Proposition 3.1]. We sketch the proof here.

*Proof.* Step 1. Solve the elliptic problem in bounded domains. Set  $\Omega_{L,k} = \Omega_L \cap \{(x_1, x_2) \mid |x_1| \leq k\}$ . It follows from [23, Theorem 12.5] that there is a solution  $\psi_{L,k} \in C^{2,\alpha}(\Omega_{L,k}) \cap C^0(\bar{\Omega}_{L,k})$  for the problem

$$\begin{cases} \left(\left(1 - \zeta^2\left(\frac{|\nabla\psi|}{\sqrt{\gamma}\check{H}_L^{\frac{\gamma+1}{2}}}\right)\right)\delta_{ij} + \zeta\left(\frac{\partial_i\psi}{\sqrt{\gamma}\check{H}_L^{\frac{\gamma+1}{2}}}\right)\zeta\left(\frac{\partial_j\psi}{\sqrt{\gamma}\check{H}_L^{\frac{\gamma+1}{2}}}\right)\right)\partial_{ij}\psi = \check{W}_L\check{H}_L^2 & \text{in } \Omega_{L,k}, \\ \psi = 0 & \text{on } \Gamma \cap \partial\Omega_{L,k}, \quad \psi = \int_0^{x_2} \rho_0 u_{0,L}(s) ds & \text{on } \partial\Omega_{L,k} \setminus \Gamma. \end{cases} \quad (3.17)$$

Step 2. Since  $\check{W}_L(s; \rho_0) \leq 0$  if  $s \leq 0$  and  $\check{W}_L(s; \rho_0) = 0$  if  $s \geq m_L$ , applying the maximum principle implies that

$$0 \leq \psi_{L,k} \leq m_L.$$

Step 3. The Hölder gradient estimate [23, Theorem 12.4] and the Hölder gradient estimate near the corners [15, Section 2.2] show that

$$\|\psi_{L,2k}\|_{C^{1,\beta}(\bar{\Omega}_{L,k})} \leq \mathfrak{C}\rho_0 \quad (3.18)$$

where  $\mathfrak{C}$  depends on  $u_0$  and  $\beta \in (0, \alpha)$  is a positive constant.

Step 4. The estimate (3.18), together with Arzela-Ascoli lemma, implies that there exists a subsequence of  $\{\psi_{L,k}\}$ , still labelled by  $\{\psi_{L,k}\}$ , converging to  $\psi_L$  in any compact subset of  $\Omega_L$ . Note that there exists a uniform constant  $C$  such that

$$\Sigma(\mathcal{B}_L(\psi_L; \rho_0)) \geq C\rho_0^{\frac{\gamma+1}{2}}.$$

Therefore, there exists a  $\check{\rho}_{0,L}$  such that if  $\rho_0 > \check{\rho}_{0,L}$ , one has

$$\frac{|\nabla\psi_L|}{\Sigma(\mathcal{B}_L(\psi_L; \rho_0))} \leq \frac{\mathfrak{C}\rho_0}{C\rho_0^{\frac{\gamma+1}{2}}} = \frac{\mathfrak{C}}{C}\rho_0^{\frac{1-\gamma}{2}} \leq \frac{1}{4}.$$

This finishes the proof of the lemma.  $\square$

The rest of this section is devoted to prove that  $\check{\rho}_{0,L}$  is indeed independent of  $L$ , so that there exist subsonic flows in  $\Omega$  as long as  $\rho_0$  is suitably large, which yields the proof of Proposition 3.1.

**3.2. Uniform estimates for subsonic flows in nozzles.** Since the stream function is unbounded as  $x_2 \rightarrow \infty$ , in order to get uniform estimates for the stream function, we study the difference of the stream function from the one in the upstream.

**Proposition 3.3.** *For any  $\epsilon > 0$ , there exists an  $\bar{L} > 0$  such that if  $L > \bar{L}$ , then for any  $\rho_0 \in ((1 + \epsilon)\rho_0^*, \infty)$ , there exists a unique triple  $(\chi(s), \rho_{1,L}, u_{1,L})$  satisfying that*

- (1)  $\rho_{1,L}$  is a constant,  $\chi$  is an increasing function maps  $[0, L]$  to  $[J, L]$ , and  $u_{1,L}$  is a function on  $[J, L]$ , where  $J = \sup f(x_1)$ ;
- (2) the triple  $(\chi(s), \rho_{1,L}, u_{1,L})$  satisfies

$$0 < \rho_{1,L} < \rho_0, \quad u_{1,L}(x_2) > 0 \quad \text{and} \quad \max_{J \leq x_2 \leq L} u_{1,L}(x_2) < \sqrt{\gamma}\rho_{1,L}^{\frac{\gamma+1}{2}}; \quad (3.19)$$

- (3) for  $s \in [0, L]$ ,  $(\chi(s), \rho_{1,L}, u_{1,L})$  satisfies

$$\frac{u_{0,L}^2(s)}{2} + h(\rho_0) = \frac{u_{1,L}^2(\chi(s))}{2} + h(\rho_{1,L}) \quad (3.20)$$

and

$$\int_0^s \rho_0 u_{0,L}(t) dt = \int_J^{\chi(s)} \rho_{1,L} u_{1,L}(t) dt. \quad (3.21)$$

Furthermore, let  $\bar{\psi}_L$  and  $\hat{\psi}_L$  be defined as follows,

$$\bar{\psi}_L(x_2) = \rho_0 \int_0^{x_2} u_{0,L}(s) ds \quad \text{and} \quad \hat{\psi}_L(x_2) = \rho_{1,L} \int_J^{x_2} u_{1,L}(s) ds. \quad (3.22)$$

Then, for  $x_2 \in [J, L]$ , it holds that

$$0 \leq \bar{\psi}(x_2) - \hat{\psi}(x_2) \leq C\rho_0, \quad (3.23)$$

where  $C$  is a uniform constant depending only on  $J$  and  $\max u_0$ .

*Proof.* Define  $\underline{\varrho}_{1,L}$  and  $\bar{\varrho}_{1,L}$  to be the constants satisfying

$$h(\bar{\varrho}_{1,L}) = \frac{1}{2} \min u_{0,L}^2(x_2) + h(\rho_0) \quad \text{and} \quad \frac{1}{2} \gamma \underline{\varrho}_{1,L}^{\gamma-1} + h(\underline{\varrho}_{1,L}) = \frac{1}{2} \max u_{0,L}^2(x_2) + h(\rho_0). \quad (3.24)$$

If  $\rho_0 > \rho_0^*$ , then it is easy to see that  $\underline{\varrho}_{1,L} < \rho_0 < \bar{\varrho}_{1,L}$ . Furthermore, for any  $\rho \in (\underline{\varrho}_{1,L}, \bar{\varrho}_{1,L})$ , one has

$$\min_{s \in [0, L]} D(s; \rho) > 0 \quad \text{and} \quad \max_{s \in [0, L]} \sqrt{D(s; \rho)} < \sqrt{\gamma} \rho^{\frac{\gamma-1}{2}} \quad \text{for } s \in [0, 1],$$

where  $D(s; \rho) = 2(h(\rho_0) - h(\rho)) + u_{0,L}^2(s)$ .

Differentiating (3.21) with respect to  $s$  and substituting (3.20) into the associated equation yield that

$$\frac{d\chi}{ds} = \frac{\rho_0 u_{0,L}(s)}{\rho_{1,L} \sqrt{D(s; \rho_{1,L})}} \quad \text{for } s \in [0, L], \text{ and } \chi(0) = J. \quad (3.25)$$

Hence, it suffices to show that there exists an  $\bar{L} > 0$  such that if  $L > \bar{L}$ , then for any  $\rho_0 \in ((1 + \epsilon)\rho_0^*, \infty)$  there exists a unique  $\rho_{1,L} \in (\underline{\varrho}_{1,L}, \bar{\varrho}_{1,L})$  such that

$$\int_0^L \frac{\rho_0 u_{0,L}(s)}{\rho_{1,L} \sqrt{D(s; \rho_{1,L})}} ds = L - J. \quad (3.26)$$

Once  $\rho_{1,L}$  is determined, then  $\chi(s)$  follows from (3.25) and  $u_1(\chi(s)) = \sqrt{D(s; \rho_{1,L})}$ .

Let  $G(\rho) = \int_0^L \frac{\rho_0 u_{0,L}(s)}{\rho \sqrt{D(s; \rho)}} ds$ . A direct computation yields that

$$G'(\rho) = \int_0^L \frac{\rho_0 u_{0,L}(s)}{\rho^2 D^{3/2}(s; \rho)} (\gamma \rho^{\gamma-1} - D(s; \rho)) ds, \quad (3.27)$$

which is always positive for  $\rho \in (\underline{\varrho}_{1,L}, \bar{\varrho}_{1,L})$ . Thus  $G(\rho)$  is a strictly increasing function in  $(\underline{\varrho}_{1,L}, \bar{\varrho}_{1,L})$ . Therefore, the range of  $G(\rho)$  on  $(\underline{\varrho}_{1,L}, \bar{\varrho}_{1,L})$  is  $(G(\underline{\varrho}_{1,L}), G(\bar{\varrho}_{1,L}))$ . We claim that  $L - J$  lies in the interval  $(G(\underline{\varrho}_{1,L}), G(\bar{\varrho}_{1,L}))$  if  $L$  is sufficiently large.

First, it is easy to see that  $G(\rho_0) = L > L - J$ . It follows from the definition of  $\underline{\varrho}_{1,L}$  in (3.24) that

$$\begin{aligned} G(\underline{\varrho}_{1,L}) &= \int_0^L \frac{\rho_0 u_{0,L}(s)}{\underline{\varrho}_{1,L} \sqrt{\gamma \underline{\varrho}_{1,L}^{\gamma-1} + u_{0,L}^2(s) - \max u_{0,L}^2(s)}} ds \\ &= \int_0^L \frac{\rho_0 u_{0,L}(s)}{\left( \frac{\gamma-1}{\gamma(\gamma+1)} \max u_{0,L}^2(s) + \frac{2}{\gamma+1} \rho_0^{\gamma-1} \right)^{\frac{1}{\gamma-1}} \sqrt{\frac{2}{\gamma+1} (\gamma \rho_0^{\gamma-1} - \max u_{0,L}^2(s)) + u_{0,L}^2(s)}} ds \\ &= \int_0^L \frac{1}{\left( \frac{\gamma-1}{\gamma+1} \frac{\max u_{0,L}^2(s)}{\gamma \rho_0^{\gamma-1}} + \frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} \sqrt{\frac{2}{\gamma+1} \left( \frac{\gamma \rho_0^{\gamma-1}}{\max u_{0,L}^2(s)} - 1 \right) \frac{\max u_{0,L}^2(s)}{u_{0,L}^2(s)} + 1}} ds. \end{aligned} \quad (3.28)$$

When  $\rho_0 > \rho_0^* = \left( \frac{\max u_{0,L}^2}{\gamma} \right)^{\frac{1}{\gamma-1}}$ , one has

$$G(\underline{\rho}_{1,L}) \leq \int_0^L \frac{1}{\left( \frac{\gamma-1}{\gamma+1} \frac{\max u_{0,L}^2(s)}{\gamma \rho_0^{\gamma-1}} + \frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} \sqrt{\frac{2}{\gamma+1} \left( \frac{\gamma \rho_0^{\gamma-1}}{\max u_{0,L}^2} - 1 \right) + 1}} ds. \quad (3.29)$$

Define

$$\mathcal{G}(\tau) = \left( \frac{\gamma-1}{\gamma+1} \tau + \frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} \sqrt{\frac{2}{\gamma+1} \left( \frac{1}{\tau} - 1 \right) + 1}.$$

The straightforward computations yield

$$\mathcal{G}'(\tau) = \left( \frac{\gamma-1}{\gamma+1} \tau + \frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} \sqrt{\frac{2}{\gamma+1} \left( \frac{1}{\tau} - 1 \right) + 1} \frac{\tau-1}{\tau(2+\tau(\gamma-1))}, \quad (3.30)$$

so  $\mathcal{G}(\tau)$  is a decreasing function on  $[0, 1]$ . Since  $\mathcal{G}(1) = 1$ , for any  $\epsilon > 0$ , there exists an  $\bar{L} > 0$  such that if  $L > \bar{L}$ , it holds that

$$\mathcal{G}\left(\frac{1}{(1+\epsilon)^{\gamma-1}}\right) \geq \frac{L}{L-J}.$$

Therefore, for any  $\rho_0 > (1+\epsilon)\rho_0^*$ , if  $L > \bar{L}$ ,

$$G(\underline{\rho}_{1,L}) \leq \int_0^L \frac{1}{\mathcal{G}\left(\frac{\max u_{0,L}^2(s)}{\gamma \rho_0^{\gamma-1}}\right)} ds \leq \int_0^L \frac{1}{\mathcal{G}\left(\frac{1}{(1+\epsilon)^{\gamma-1}}\right)} ds \leq \int_0^L \frac{L-J}{L} ds = L-J. \quad (3.31)$$

Hence, for any  $\rho_0 \in ((1+\epsilon)\rho_0^*, \infty)$ , there exists a unique  $\rho \in (\underline{\rho}_{1,L}, \rho_0)$  satisfying  $G(\rho) = L-J$ .

Define  $\bar{\chi}(s) = \chi(s) - s$ . It follows from (3.25) that

$$\bar{\chi}'(s) = \frac{\rho_0 u_{0,L}(s) - \rho_{1,L} u_{1,L}(\chi(s))}{\rho_{1,L} u_{1,L}(\chi(s))}. \quad (3.32)$$

As mentioned in Section 2, for given  $\mathfrak{s}$  in (2.14),  $\mathfrak{m}$  is strictly decreasing with respect to  $\rho$  in the subsonic region. Since  $\rho_{1,L} < \rho_0$ , one has  $\rho_0 u_{0,L}(s) < \rho_{1,L} u_{1,L}(\chi(s))$ . Therefore,  $\bar{\chi}'(s) < 0$ . Hence,  $\bar{\chi}(L) \leq \bar{\chi}(s) \leq \bar{\chi}(0)$ . This gives  $0 \leq \chi(s) - s \leq J$  which is equivalent to  $0 \leq x_2 - \chi^{-1}(x_2) \leq J$  for  $x_2 \in [J, L]$ . If  $x_2 \in [J, L]$ , then direct computations yield

$$\begin{aligned} \bar{\psi}(x_2) - \hat{\psi}(x_2) &= \int_0^{x_2} \rho_0 u_{0,L}(s) ds - \int_J^{x_2} \rho_{1,L} u_{1,L}(s) ds \\ &= \int_0^{x_2} \rho_0 u_{0,L}(s) ds - \int_{\chi^{-1}(J)}^{\chi^{-1}(x_2)} \rho_{1,L} u_{1,L}(\chi(s)) \chi'(s) ds \\ &= \int_0^{x_2} \rho_0 u_{0,L}(s) ds - \int_0^{\chi^{-1}(x_2)} \rho_{0,L} u_{0,L}(s) ds \\ &= \int_{\chi^{-1}(x_2)}^{x_2} \rho_0 u_{0,L}(s) ds, \end{aligned} \quad (3.33)$$

where (3.25) has been used in the third equality. Since  $0 \leq x_2 - \chi^{-1}(x_2) \leq J$ , one gets

$$0 \leq \bar{\psi}(x_2) - \hat{\psi}(x_2) \leq C \rho_0, \quad (3.34)$$

where the constant  $C$  depends only on  $J$  and  $\max u_0$ . This finishes the proof for the proposition.  $\square$

One of our key estimates is the following upper and lower bounds estimate for  $\psi_L$ , which also plays an important role in proving the uniform gradient estimate for  $\psi_L - \bar{\psi}_L$ .

**Proposition 3.4.** *Let  $\hat{\Omega}_L = \{(x_1, x_2) | x_2 \in (J, L), x_1 \in \mathbb{R}\}$ . For any  $\epsilon > 0$ , there exists an  $\bar{L} > 0$  such that if  $L > \bar{L}$ , then for any  $\rho_0 \in ((1 + \epsilon)\rho_0^*, \infty)$ , if  $\psi_L$  is a subsonic solution of the problem (3.9), then it holds that*

$$\psi_L \geq \hat{\psi}_L \text{ in } \hat{\Omega}_L \text{ and } 0 \leq \psi_L \leq \bar{\psi}_L \text{ in } \Omega_L, \quad (3.35)$$

where  $\hat{\psi}_L$  and  $\bar{\psi}_L$  are defined in (3.22).

*Proof.* Define

$$\Psi_L(x) = \psi_L(x) - \bar{\psi}_L(x_2) \text{ for } x \in \Omega \text{ and } \hat{\Psi}_L = \hat{\psi}_L - \psi_L \text{ for } x \in \hat{\Omega}_L. \quad (3.36)$$

Note that  $\hat{\psi}_L$  satisfies the equation (3.9) with the boundary conditions  $\hat{\psi}_L = 0$  at  $x_2 = J$  and  $\hat{\psi}_L = m_L$  at  $x_2 = L$ . Then straightforward computations show that  $\hat{\Psi}_L$  solves the following problem

$$\begin{cases} \partial_i \left( a_{ij}(\nabla \psi, \psi; \nabla \hat{\psi}, \hat{\psi}) \partial_j \hat{\Psi}_L + b_i(\nabla \psi, \psi; \nabla \hat{\psi}, \hat{\psi}) \hat{\Psi}_L \right) \\ = b_i(\nabla \psi, \psi; \nabla \hat{\psi}, \hat{\psi}) \partial_i \hat{\Psi}_L + d(\nabla \psi, \psi; \nabla \hat{\psi}, \hat{\psi}) \hat{\Psi}_L \quad \text{in } \hat{\Omega}_L, \\ \hat{\Psi}_L \leq 0 \text{ if } x_2 = J, \text{ and } \hat{\Psi}_L = 0 \text{ if } x_2 = L, \end{cases} \quad (3.37)$$

where the Einstein summation convention is used and  $a_{ij}$ ,  $b_i$ , and  $d$  are defined as follows,

$$a_{ij}(\nabla \psi, \psi; \nabla \hat{\psi}, \hat{\psi}) = \int_0^1 \frac{1}{H_L(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t})} \left( \delta_{ij} + \frac{\partial_i \hat{\psi}_{L,t} \partial_j \hat{\psi}_{L,t}}{\gamma H_L^{\gamma+1}(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t}) - |\nabla \hat{\psi}_{L,t}|^2} \right) dt, \quad (3.38)$$

$$b_i(\nabla \psi, \psi; \nabla \hat{\psi}, \hat{\psi}) = - \int_0^1 \frac{\partial_i \hat{\psi}_{L,t} F_L(\hat{\psi}_{L,t}) F'_L(\hat{\psi}_{L,t}) H_L(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t})}{\gamma H_L^{\gamma+1}(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t}) - |\nabla \hat{\psi}_{L,t}|^2} dt, \quad (3.39)$$

and

$$\begin{aligned} d(\nabla \psi, \psi; \nabla \hat{\psi}, \hat{\psi}) &= \int_0^1 \frac{H_L^3(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t}) \left( F_L(\hat{\psi}_{L,t}) F'_L(\hat{\psi}_{L,t}) \right)^2}{\gamma H_L^{\gamma+1}(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t}) - |\nabla \hat{\psi}_{L,t}|^2} dt \\ &\quad + \int_0^1 H_L(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t}) \left( F_L(\hat{\psi}_{L,t}) F''_L(\hat{\psi}_{L,t}) + (F'_L(\hat{\psi}_{L,t}))^2 \right) dt \\ &= \sum_{i=1}^2 d_i(\nabla \psi, \psi; \nabla \hat{\psi}, \hat{\psi}), \end{aligned} \quad (3.40)$$

for  $i, j = 1, 2$ , with  $\hat{\psi}_{L,t} = t\hat{\psi}_L + (1-t)\psi_L$  for  $t \in [0, 1]$ . Here and in what follows we neglect the parameter  $\rho_0$  in the coefficients when there is no confusion.

Set  $\hat{\Psi}_L^+ = \max\{\hat{\Psi}_L, 0\}$ . Multiplying the equation in (3.37) with  $\hat{\Psi}_L^+$  and integrating by parts imply that

$$\begin{aligned} & \iint_{\hat{\Omega}_L} \left[ \left| \nabla \hat{\Psi}_L^+ \right|^2 \int_0^1 \frac{1}{H_L(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t})} dt \right. \\ & \quad \left. + \int_0^1 \frac{\left( H_L^2(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t}) F_L(\hat{\psi}_{L,t}) F_L'(\hat{\psi}_{L,t}) \hat{\Psi}_L^+ - \nabla \hat{\psi} \cdot \nabla \hat{\Psi}_L^+ \right)^2}{H_L(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t}) (\gamma H_L^{\gamma+1}(|\nabla \hat{\psi}_{L,t}|^2, \hat{\psi}_{L,t}) - |\nabla \hat{\psi}_{L,t}|^2)} dt \right] dx_1 dx_2 \\ & = - \iint_{\hat{\Omega}_L} d_2(\nabla \psi, \psi; \nabla \hat{\psi}, \hat{\psi}) \left( \hat{\Psi}_L^+ \right)^2 dx_1 dx_2. \end{aligned} \quad (3.41)$$

Then direct computations yield that

$$F_L(\hat{\psi}_{L,t}) F_L''(\hat{\psi}_{L,t}) + \left( F_L'(\hat{\psi}_{L,t}) \right)^2 = \frac{u_{0,L}''(\kappa_L(\hat{\psi}_{L,t}; \rho_0))}{\rho_0 u_{0,L}(\kappa_L(\hat{\psi}_{L,t}; \rho_0))} \geq 0,$$

hence  $d_2 \geq 0$  for subsonic flows with the assumption (1.7). Therefore, it follows from (3.41) that

$$\nabla \hat{\Psi}_L^+ = 0 \quad \text{in } \hat{\Omega}_L.$$

Since  $\hat{\Psi}_L^+ = 0$  on  $\partial \hat{\Omega}_L$ , one has  $\hat{\Psi}_L^+ = 0$  in  $\hat{\Omega}_L$ . Thus  $\hat{\Psi}_L \leq 0$  in  $\hat{\Omega}_L$  which is equivalent to  $\psi_L \geq \hat{\psi}_L$  in  $\hat{\Omega}_L$ .

Similarly, one can show that  $\psi_L \leq \bar{\psi}_L$  in  $\Omega_L$ . This finishes the proof of the proposition.  $\square$

**Corollary 3.5.** *For any  $\epsilon > 0$ , there exists an  $\bar{L} > 0$  such that if  $L > \bar{L}$ , then for any  $\rho_0 \in ((1+\epsilon)\rho_0^*, \infty)$ , if  $\psi_L$  is a subsonic solution of the problem (3.9) satisfying  $0 \leq \psi_L \leq m_L$ , then it holds that*

$$-C\rho_0 \leq \Psi_L \leq 0 \quad (3.42)$$

where the constant  $C$  depends only on  $J$  and  $\max u_0$ .

*Proof.* First, it follows from (3.35) that  $\Psi_L \leq 0$ . If  $x_2 \geq J$ , then

$$\Psi_L = \psi_L - \bar{\psi}_L \geq \hat{\psi}_L - \bar{\psi}_L \geq -C\rho_0,$$

where (3.23) has been used. If  $x_2 \in (0, J)$ , then (3.35) implies that  $\psi_L \geq 0$ . Therefore,

$$\Psi_L \geq \psi_L - \bar{\psi}_L \geq -\bar{\psi}_L \geq -C\rho_0.$$

This proves the corollary.  $\square$

As a consequence of the elliptic estimate, one has the following lemma.

**Lemma 3.6.** *For any  $\epsilon > 0$ , there exists an  $\bar{L} > 0$  such that if  $L > \bar{L}$ , then for any  $\rho_0 \in ((1+\epsilon)\rho_0^*, \infty)$ , if  $\psi_L$  is a subsonic solution of the problem (3.9) satisfying  $0 \leq \psi_L \leq m_L$ , then*

$$\|\Psi_L\|_{C^{1,\beta}(\overline{\Omega}_L)} \leq \mathcal{C}\rho_0 \quad (3.43)$$

with  $\beta \in (0, \alpha)$  independent of  $L$ .

*Proof.* Note that  $\Psi_L$  satisfies the problem

$$\begin{cases} \left( \left( 1 - \frac{|\nabla \psi_L|^2}{\gamma H_L^{\gamma+1}} \right) \delta_{ij} + \frac{\partial_i \psi_L \partial_j \psi_L}{\gamma H_L^{\gamma+1}} \right) \partial_{ij} \Psi_L = F_L F'_L (H_L^2 - \rho_0^2 - \frac{(\partial_1 \psi_L)^2}{\gamma H_L^{\gamma+1}} \rho_0^2) & \text{in } \Omega_L, \\ \Psi_L = - \int_0^{f(x_1)} \rho_0 u_{0,L}(s) ds & \text{on } \Gamma, \quad \Psi_L = 0 & \text{on } \Gamma_L. \end{cases} \quad (3.44)$$

It follows from the Hölder gradient estimate [23, Theorem 12.4] and the estimate near the corners [15, Section 2.2] for the elliptic equations that (3.43) holds.  $\square$

In fact, we can also prove the following far field behavior and uniform integral estimate for  $\Psi_L$ .

**Lemma 3.7.** *Suppose that  $u_0''(x_2) \geq 0$ . If  $\psi_L$  is a subsonic solution of the problem (3.9), then we have*

$$|\psi_L - \bar{\psi}_L| \rightarrow 0 \quad \text{uniformly with respect to } x_2 \in [0, L] \text{ as } |x_1| \rightarrow \infty \quad (3.45)$$

and

$$\|\nabla(\psi_L - \bar{\psi}_L)\|_{L^2(\Omega_L)} \leq \mathcal{C} \quad (3.46)$$

where the constant  $\mathcal{C}$  depends on elliptic coefficients and is independent of  $L$ . Furthermore, the solution of the problem (3.9) is unique.

*Proof.* First, the far field behavior (3.45) and the uniqueness of the solution follow from [16, Lemma 4.1] and [40, Proposition 5].

Note that  $\Psi_L$  satisfies the following problem

$$\begin{cases} \partial_i (a_{ij}(\nabla \psi, \psi; \nabla \bar{\psi}, \bar{\psi}) \partial_j \Psi_L + b_i(\nabla \psi, \psi; \bar{\psi}, \bar{\psi}) \Psi_L) \\ = b_i(\nabla \psi, \psi; \nabla \bar{\psi}, \bar{\psi}) \partial_i \Psi_L + d(\nabla \psi, \psi; \nabla \bar{\psi}, \bar{\psi}) \Psi_L & \text{in } \Omega_L, \\ \Psi_L = -\bar{\psi}_L \text{ if } x_2 = f(x_1), \text{ and } \Psi_L = 0 \text{ if } x_2 = L, \end{cases} \quad (3.47)$$

where the coefficients  $a_{ij}$ ,  $b_i$ ,  $d$  are defined as in (3.38)-(3.40).

Multiplying the equation in (3.47) with  $\Psi_L$  and integrating in the domain  $\Omega_{L,N} = \Omega_L \cap \{|x_1| \leq N\}$  yield that

$$\begin{aligned}
& \iint_{\Omega_{L,N}} (a_{ij} \partial_i \Psi_L \partial_j \Psi_L + 2b_i \Psi_L \partial_i \Psi_L + d_1 \Psi_L^2) dx_1 dx_2 \\
&= \int_{\partial\Omega_{L,N}} (a_{ij} \partial_j \Psi_L + b_i \Psi_L) \Psi_L n_i dS - \iint_{\Omega_{L,N}} d_2 \Psi_L^2 dx_1 dx_2 \\
&= - \int_{\{x_2=f(x_1)\}} (a_{ij} \partial_j \Psi_L + b_i \Psi_L) \bar{\Psi}_L n_i dS - \iint_{\Omega_{L,N}} d_2 \Psi_L^2 dx_1 dx_2 \\
&\quad - \int_{\{x_1=N\}} (a_{ij} \partial_j \Psi_L + b_i \Psi_L) \Psi_L n_i dS + \int_{\{x_1=-N\}} (a_{ij} \partial_j \Psi_L + b_i \Psi_L) \Psi_L n_i dS
\end{aligned} \tag{3.48}$$

where  $n = (n_1, n_2)$  is the unit normal to the boundary  $\partial\Omega_{L,N}$ . Thus

$$\begin{aligned}
& \iint_{\Omega_{L,N}} (a_{ij} \partial_i \Psi_L \partial_j \Psi_L + 2b_i \Psi_L \partial_i \Psi_L + d_1 \Psi_L^2) dx_1 dx_2 \\
&\leq \mathcal{C} \left( \int_{\mathbb{R}} |\bar{\psi}_L(f(x_1))| dx_1 + \int_{f(N)}^L |\Psi_L(N, x_2)| dx_2 + \int_{f(-N)}^L |\Psi_L(-N, x_2)| dx_2 \right).
\end{aligned} \tag{3.49}$$

Here one has used the uniform bounds for  $C^1$ -norm of  $\Psi_L$  and the positivity of  $d_2$ .

Then direct computations give

$$\begin{aligned}
& a_{ij} \partial_i \Psi_L \partial_j \Psi_L + 2b_i \Psi_L \partial_i \Psi_L + d_1 \Psi_L^2 \\
&= \int_0^1 \frac{1}{H_L(|\nabla \bar{\psi}_{L,t}|^2, \bar{\psi}_{L,t})} \left( |\nabla \Psi_L|^2 + \frac{(\nabla \psi_t \cdot \nabla \Psi_L - FF' \Psi_L H_L^2(|\nabla \bar{\psi}_{L,t}|^2, \bar{\psi}_{L,t}))^2}{\gamma H_L^{\gamma+1}(|\nabla \bar{\psi}_{L,t}|^2, \bar{\psi}_{L,t}) - |\nabla \psi_t|^2} \right) dt,
\end{aligned} \tag{3.50}$$

where  $\bar{\psi}_{L,t} = t\bar{\psi}_L + (1-t)\psi_L$  for  $t \in [0, 1]$ . Substituting (3.50) into (3.49) yields that for any  $N > 0$ , one has

$$\begin{aligned}
& \iint_{\Omega_{L,N}} |\nabla \Psi_L|^2 dx_1 dx_2 \leq \mathcal{C} \left( \int_{\mathbb{R}} |\bar{\psi}_L(x_1, f(x_1))| dx_1 + \int_{f(N)}^L |\Psi_L(N, x_2)| dx_2 \right. \\
&\quad \left. + \int_{f(-N)}^L |\Psi_L(-N, x_2)| dx_2 \right).
\end{aligned} \tag{3.51}$$

The asymptotic behavior (3.45) implies that the second term on the right-hand side of (3.51) tends to zero as  $N$  goes to infinity. Moreover, thanks to the asymptotic assumption to the nozzle wall  $x_2 = f(x_1)$ , the first term on the right-hand side of (3.51) is uniformly bounded (independent of  $L$ ). Hence, we have

$$\iint_{\Omega_L} |\nabla \Psi_L|^2 dx_1 dx_2 \leq \mathcal{C} \int_{\mathbb{R}} |\bar{\psi}_L(f(x_1))| dx_1 \leq \mathcal{C}. \tag{3.52}$$

This finishes the proof of the lemma.  $\square$



Given  $\rho_0 \in (\rho_0^*, \infty)$ , let  $\mathcal{S}_L(\rho_0)$  be the set of all solutions of the problem (3.9) associated with  $\rho_0$ . Define

$$\mathcal{M}_L(\rho_0) = \sup_{\psi_L \in \mathcal{S}_L(\rho_0)} \sup_{x \in \Omega_L} \frac{|\nabla \psi_L(x; \rho_0)|}{\Sigma(\mathcal{B}_L(\psi_L; \rho_0))}. \quad (3.53)$$

Set

$$\bar{\rho}_{0,L} = \inf \left\{ s \mid \text{for any } \rho_0 > s, \mathcal{M}_L(\rho_0) < \frac{1}{4} \right\} \quad \text{and} \quad \rho^{**} = (20 \max u_0^2)^{\frac{1}{\gamma-1}}. \quad (3.54)$$

**Lemma 3.8.** *If  $\bar{\rho}_{0,L} > \rho^{**}$ , then  $\mathcal{M}(\bar{\rho}_{0,L}) = \frac{1}{4}$ .*

*Proof.* First, it follows from the continuous dependence on the parameters for solutions of uniformly elliptic equations that  $\mathcal{M}_L(\bar{\rho}_{0,L}) \leq \frac{1}{4}$ . If  $\mathcal{M}_L(\bar{\rho}_{0,L}) < 1/4$  and  $\bar{\rho}_{0,L} > \rho^{**}$ , then it is easy to see that there exists a  $\delta > 0$  such that  $\mathcal{M}_L(s) \leq 1/4$  for  $s \in (\bar{\rho}_{0,L} - \delta, \bar{\rho}_{0,L})$ . There is a contradiction. So the proof of the lemma is completed.  $\square$

Now we have the following lemma.

**Lemma 3.9.** *There exists a  $\bar{\rho}_0 \in (\rho_0^*, \infty)$  independent of  $L$  such that if  $\rho_0 > \bar{\rho}_0$ , there exists a subsonic solution of (3.9) satisfying*

$$\frac{|\nabla \psi_L|}{\sqrt{\gamma} H_L^{\frac{\gamma+1}{2}} (|\nabla \psi_L|^2, \psi_L)} \leq \frac{1}{4}. \quad (3.55)$$

*Proof.* If  $\bar{\rho}_{0,L} > \rho^{**}$ , then it follows from Lemma 3.8 that the problem (3.9) associated with  $\rho_0 = \bar{\rho}_{0,L}$  has a solution  $\psi_L$  satisfying

$$\sup_{\Omega_L} \frac{|\nabla \psi_L|}{\Sigma(\mathcal{B}_L(\psi_L; \rho_0))} = \frac{1}{4}. \quad (3.56)$$

It follows from Lemma 3.6 and the definition of  $\mathcal{B}_L$  that one has

$$\frac{1}{4} \leq \frac{\sup |\nabla \psi_L|}{\inf \Sigma(\mathcal{B}(\psi_L; \rho_0))} \leq \frac{\mathcal{C} \bar{\rho}_{0,L}}{C \bar{\rho}_{0,L}^{\frac{\gamma+1}{2}}} = \mathcal{C}^\sharp \rho_0^{\frac{1-\gamma}{2}}, \quad (3.57)$$

where  $\mathcal{C}^\sharp$  is a constant independent of  $L$ . Therefore,

$$\bar{\rho}_{0,L} \leq (4 \mathcal{C}^\sharp)^{\frac{2}{\gamma-1}}. \quad (3.58)$$

Choose

$$\bar{\rho}_0 = \max \left( \rho^{**}, (4 \mathcal{C}^\sharp)^{\frac{2}{\gamma-1}} \right). \quad (3.59)$$

It is easy to see that  $\bar{\rho}_0$  is independent of  $L$  and that if  $\rho_0 > \bar{\rho}_0$ , the problem (3.9) has a solution  $\psi_L$  satisfying  $\frac{|\nabla \psi_L|}{\Sigma(\mathcal{B}(\psi_L; \rho_0))} \leq 1/4$ . This finishes the proof of the lemma.  $\square$

**3.3. Existence of subsonic solutions with large incoming density.** If  $\rho_0 > \bar{\rho}_0$ , then the problem (3.9) has a solution  $\psi_L$  satisfying

$$\|\psi_L - \bar{\psi}_L\|_{C^1(\Omega)} \leq \mathcal{C}\rho_0 \quad \text{and} \quad \|\nabla(\psi_L - \bar{\psi}_L)\|_{L^2(\Omega)} \leq \mathcal{C}, \quad \text{and} \quad \sup_{\Omega_L} \frac{|\nabla\psi_L|}{\Sigma(\mathcal{B}_L(\psi_L; \rho_0))} \leq \frac{1}{4}$$

where  $\mathcal{C}$  is a constant independent of  $L$ . Let  $L \rightarrow \infty$ , then there exists a subsequence of  $\{\psi_L\}$  still labelled by  $\{\psi_L\}$ , converging to  $\psi$  satisfying

$$\|\psi - \bar{\psi}\|_{C^1(\Omega)} \leq \mathcal{C}\rho_0 \quad \text{and} \quad \|\nabla(\psi - \bar{\psi})\|_{L^2(\Omega)} \leq \mathcal{C}, \quad \text{and} \quad \sup_{\Omega} \frac{|\nabla\psi|}{\Sigma(\mathcal{B}(\psi; \rho_0))} \leq \frac{1}{4} \quad (3.60)$$

where the constant  $\mathcal{C}$  is a uniform constant. Hence  $\psi$  is a subsonic solution of the problem (2.17). This finishes the proof for Proposition 3.1.

#### 4. FINE PROPERTIES OF THE SUBSONIC SOLUTIONS PAST A WALL

In this section, we study properties of subsonic solutions for (2.17) constructed in previous section, such as asymptotic behaviors and positivity of horizontal velocity of subsonic flows.

**4.1. Asymptotic behavior at the far fields.** We claim that

$$|\nabla\Psi(x_1, x_2)| \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty, \quad (4.1)$$

where  $\Psi = \psi - \bar{\psi}$  with  $\bar{\psi}$  defined in (3.2). Note that the estimate (4.1) gives the asymptotic behavior of subsonic flows in the far fields. It follows from the definition that  $\Psi$  satisfies the equation

$$\left( \left( 1 - \frac{|\nabla\psi|^2}{\gamma H^{\gamma+1}} \right) \delta_{ij} + \frac{\partial_i \psi \partial_j \psi}{\gamma H^{\gamma+1}} \right) \partial_{ij} \Psi = FF'(H^2 - \rho_0^2 - \frac{(\partial_1 \psi)^2}{\gamma H^{\gamma+1}} \rho_0^2) \quad \text{in } \Omega. \quad (4.2)$$

Since  $\|\Psi\|_{L^\infty(\bar{\Omega})} \leq \mathcal{C}\rho_0$ , it follows from the Hölder gradient estimate that

$$\|\Psi\|_{C^{1,\beta}(\bar{\Omega})} \leq \mathcal{C}\rho_0. \quad (4.3)$$

Now we prove (4.1) by contradiction argument. If (4.1) is false, for any positive constant  $\varepsilon_0$ , there exists a sequence  $\{x^{(i)} = (x_1^{(i)}, x_2^{(i)})\}_{i=1}^\infty$  going to infinity such that  $|\nabla\Psi(x_1^{(i)}, x_2^{(i)})| \geq \varepsilon_0$  for some positive constant  $\varepsilon_0$ . It follows from (4.3) that there exists a uniform constant  $\delta_0 > 0$  such that

$$|\nabla\Psi(x)| \geq \varepsilon_0/2 \quad \text{for any} \quad x \in B_{\delta_0}(x^{(i)}),$$

and

$$B_{\delta_0}(x^{(i)}) \cap B_{\delta_0}(x^{(j)}) = \emptyset \quad \text{for} \quad i \neq j.$$

Furthermore, it holds that

$$\iint_{\bigcup_i B_{\delta_0}(x^{(i)})} |\nabla\Psi|^2 dx_1 dx_2 = \sum_{i=1}^{+\infty} \iint_{B_{\delta_0}(x^{(i)})} |\nabla\Psi|^2 dx_1 dx_2 = \infty, \quad (4.4)$$

which contradicts to (3.46).

**4.2. Positivity of horizontal velocity except at the corners.** It follows from Proposition 2.1 that the flow  $(\rho, u, v) = (H, \frac{\partial_2 \psi}{H}, -\frac{\partial_1 \psi}{H})$  is indeed a solution of two dimensional steady Euler equations (1.1) with boundary conditions (1.4)-(1.6), if it satisfies (2.3) and (2.4). In order to justify (2.3) and (2.4), now we need only to show that the horizontal velocity of the subsonic solution is always positive except at the corner points, namely  $\partial_{x_2} \psi > 0$  in  $\bar{\Omega} \setminus \{P_1, P_2\}$ .

Since

$$\partial_{x_2} \psi = \partial_{x_2} \Psi + \bar{\psi}'(x_2) = \partial_{x_2} \Psi + \rho_0 u_0(x_2)$$

and  $|\nabla \Psi| \rightarrow 0$  as  $|x| \rightarrow \infty$ , there exist  $R > 0$  and  $\delta > 0$  such that  $\partial_{x_2} \psi > \delta$  for  $|x| > R$ . Hence, the horizontal velocity is positive for  $\{(x_1, x_2) | |x_1| \geq R, x_2 > R\}$ . Let  $\Omega' = \{(x_1, x_2) \in \mathbb{R}^2 | f(x_1) < x_2 < R, |x_1| < R\}$ . Combining the argument [40, Lemma2] and the convexity condition in (1.7) which implies

$$F(\psi)F''(\psi) + (F'(\psi))^2 \geq 0,$$

yields the positivity of the horizontal velocity for subsonic flows in bounded domain  $\Omega'$ . Consequently, the horizontal velocity of the subsonic solutions is positive in the whole domain  $\Omega$ . Thus  $(\rho, u, v)$  is a solution of the Euler system (1.1) with boundary conditions (1.4)-(1.6).

## 5. THE UNIQUENESS OF SUBSONIC EULER FLOWS PAST A WALL

In this section, we show the uniqueness of subsonic solutions satisfying (3.46) and the asymptotic behavior (1.11)-(1.14) in the far fields.

Assume that  $\psi_{(i)} \in C^{2,\alpha}(\Omega) \cap C^{1,\beta}(\bar{\Omega})$  ( $i = 1, 2$ ) solve the problem (2.17) and satisfy

$$\psi_{(i)} - \bar{\psi} \in L^\infty(\Omega), \quad \nabla(\psi_{(i)} - \bar{\psi}) \in L^2(\Omega), \quad \text{and} \quad |\nabla \psi_{(i)}|^2 \leq (1 - 2\varepsilon_0)\Sigma^2(\mathcal{B}(\psi_{(i)}, \rho_0)) \quad (5.1)$$

for some  $\varepsilon_0 > 0$ . Set  $\phi = \psi_{(1)} - \psi_{(2)}$ . Then it follows from (3.42) and (3.60) that  $\phi$  satisfies

$$\|\phi\|_{L^\infty(\Omega)} \leq C\rho_0 \quad \text{and} \quad \|\nabla \phi\|_{L^2(\Omega)} \leq \mathcal{C}. \quad (5.2)$$

Furthermore,  $\phi$  solves the following problem

$$\left\{ \begin{array}{l} \partial_i (a_{ij}(\nabla \psi_{(1)}, \psi_{(1)}; \nabla \psi_{(2)}, \psi_{(2)}) \partial_j \phi) + \partial_i (b_i(\nabla \psi_{(1)}, \psi_{(1)}; \nabla \psi_{(2)}, \psi_{(2)}) \phi) \\ \quad = b_i(\nabla \psi_{(1)}, \psi_{(1)}; \nabla \psi_{(2)}, \psi_{(2)}) \partial_i \phi + d(\nabla \psi_{(1)}, \psi_{(1)}; \nabla \psi_{(2)}, \psi_{(2)}) \phi \quad \text{in } \Omega, \\ \phi = 0 \quad \text{on } x_2 = f(x_1), \end{array} \right. \quad (5.3)$$

where  $a_{ij}$ ,  $b_i$ , and  $d$  are defined in (3.38)-(3.40) with  $H_L$  replaced by  $H$ . In this section, we denote  $H(t) = H(|\nabla \psi_{(1+t)}|^2, \psi_{(1+t)}; \rho_0)$  with  $\psi_{(1+t)} = (1-t)\psi_{(1)} + t\psi_{(2)}$  for  $t \in [0, 1]$ .

Denote  $B_r^\Omega(0) = B_r(0) \cap \Omega$  for  $r > 0$  and let  $\eta$  be the smooth cut-off function satisfying

$$\eta = \begin{cases} 1 & \text{for } (x_1, x_2) \in B_R^\Omega(0), \\ 0 & \text{for } (x_1, x_2) \in \Omega \setminus B_{2R}(0), \end{cases} \quad (5.4)$$

and  $|\nabla \eta| \leq 2/R$ .

Multiplying the both sides of the equation in (5.3) with  $\eta^2 \phi$  and integrating in  $\Omega$  yield that

$$\begin{aligned} & \iint_{B_{2R}^\Omega(0)} \eta^2 \left[ \int_0^1 \frac{|\nabla \phi|^2}{H(t)} + \frac{|\nabla \phi \cdot \nabla \psi_{(1+t)} - F(\psi_{(1+t)})F'(\psi_{(1+t)})H_{(t)}^2 \phi|^2}{H(t)(\gamma H_{(t)}^{\gamma+1} - |\nabla \psi_{(1+t)}|^2)} dt \right] dx_1 dx_2 \\ & + \iint_{B_{2R}^\Omega(0)} \eta^2 \left[ \int_0^1 \phi^2 \int_0^1 H(t) (F(\psi_{(1+t)})F''(\psi_{(1+t)}) + (F'(\psi_{(1+t)}))^2) dt \right] dx_1 dx_2 \\ = & -2 \iint_{B_{2R}^\Omega(0)} \left( a_{ij} \eta \partial_i \eta \phi \partial_j \phi - \eta \partial_i \eta \phi \int_0^1 \frac{\partial_i \psi_{(1+t)} F(\psi_{(1+t)}) F'(\psi_{(1+t)}) H_{(t)}^2}{(\gamma H_{(t)}^{\gamma+1} - |\nabla \psi_{(1+t)}|^2)} dt \right) dx_1 dx_2. \end{aligned} \quad (5.5)$$

Note

$$\int_0^1 H(t) (F(\psi_{(1+t)})F''(\psi_{(1+t)}) + (F'(\psi_{(1+t)}))^2) dt = \int_0^1 \frac{u_0''(\kappa(\psi_{(1+t)}; \rho_0))}{\rho_0^2 u_0(\kappa(\psi_{(1+t)}; \rho_0))} H(t) dt \geq 0. \quad (5.6)$$

Therefore, it follows from (5.5) that one has

$$\begin{aligned} & \iint_{B_{2R}^\Omega(0)} \eta^2 \left[ \int_0^1 \frac{|\nabla \phi \cdot \nabla \psi_{(1+t)} - F(\psi_{(1+t)})F'(\psi_{(1+t)})H_{(t)}^2 \phi|^2}{H(t)(H_{(t)}^2 c^2(H(t)) - |\nabla \psi_{(1+t)}|^2)} dt \right] dx_1 dx_2 \\ & \leq \mathcal{C} \iint_{B_{2R}^\Omega(0)} \eta \left( |\phi| |\nabla \eta \cdot \nabla \phi| + |\nabla \eta| \int_0^1 |F(\psi_{(1+t)})F'(\psi_{(1+t)})\phi| dt \right) dx_1 dx_2. \end{aligned} \quad (5.7)$$

Denote  $A_{R,2R}^\Omega(0) = B_{2R}^\Omega(0) \setminus \overline{B_R^\Omega(0)}$ . Noting that  $\phi$  and  $\psi_{(1+t)}$  are uniformly bounded and using the Young inequality gives

$$\begin{aligned}
& \iint_{B_{2R}^\Omega(0)} \eta^2 \left( \int_0^1 |F(\psi_{(1+t)})F'(\psi_{(1+t)})\phi|^2 dt \right) dx_1 dx_2 \\
& \leq \iint_{B_{2R}^\Omega(0)} \eta^2 \left[ \int_0^1 \frac{|\nabla \phi \cdot \nabla \psi_{(1+t)}|^2}{H_{(t)}(\gamma H_{(t)}^{\gamma+1} - |\nabla \psi_{(1+t)}|^2)} dt \right] dx_1 dx_2 \\
& \quad + \iint_{B_{2R}^\Omega(0)} \eta^2 \left[ \int_0^1 \frac{|\nabla \phi \cdot \nabla \psi_{(1+t)} - F(\psi_{(1+t)})F'(\psi_{(1+t)})H_{(t)}^2 \phi|^2}{H_{(t)}(\gamma H_{(t)}^{\gamma+1} - |\nabla \psi_{(1+t)}|^2)} dt \right] dx_1 dx_2 \\
& \leq \mathcal{C} \iint_{B_{2R}^\Omega(0)} |\nabla \phi|^2 dx_1 dx_2 + \mathcal{C} \iint_{A_{R,2R}^\Omega(0)} |\nabla \eta \cdot \nabla \phi| dx_1 dx_2 \\
& \quad + \mathcal{C} \delta \iint_{B_{2R}^\Omega(0)} \eta^2 \left( \int_0^1 |F(\psi_{(1+t)})F'(\psi_{(1+t)})\phi| dt \right)^2 dx_1 dx_2 + \mathcal{C}(\delta) \iint_{A_{R,2R}^\Omega(0)} |\nabla \eta|^2 dx_1 dx_2 \\
& \leq \mathcal{C} \iint_{B_{2R}^\Omega(0)} |\nabla \phi|^2 dx_1 dx_2 + \mathcal{C} \delta \iint_{B_{2R}^\Omega(0)} \eta^2 \int_0^1 |F(\psi_{(1+t)})F'(\psi_{(1+t)})\phi|^2 dt dx_1 dx_2 \\
& \quad + \mathcal{C}(\delta) \iint_{A_{R,2R}^\Omega(0)} |\nabla \eta|^2 dx_1 dx_2.
\end{aligned} \tag{5.8}$$

Choosing a suitable small  $\delta$  yields that

$$\begin{aligned}
& \iint_{B_R^\Omega(0)} \int_0^1 |F(\psi_{(1+t)})F'(\psi_{(1+t)})\phi|^2 dt dx_1 dx_2 \\
& \leq \mathcal{C} \iint_{B_{2R}^\Omega(0)} |\nabla \phi|^2 dx_1 dx_2 + \mathcal{C} \iint_{A_{R,2R}^\Omega(0)} \frac{1}{R^2} dx_1 dx_2 \\
& \leq \mathcal{C}.
\end{aligned} \tag{5.9}$$

It follows from (5.5) that one has

$$\begin{aligned}
& \iint_{B_R^\Omega(0)} |\nabla \phi|^2 dx_1 dx_2 \\
& \leq \mathcal{C} \iint_{A_{R,2R}^\Omega(0)} |\nabla \eta| \left( |\nabla \phi| + \phi \int_0^1 F(\psi_{(1+t)})F'(\psi_{(1+t)}) dt \right) dx_1 dx_2 \\
& \leq \mathcal{C} \|\nabla \eta\|_{L^2(A_{R,2R}^\Omega(0))} \left( \|\nabla \phi\|_{L^2(A_{R,2R}^\Omega(0))} + \left\| \phi \int_0^1 F(\psi_{(1+t)})F'(\psi_{(1+t)}) dt \right\|_{L^2(A_{R,2R}^\Omega(0))} \right).
\end{aligned} \tag{5.10}$$

In view of (5.9) and letting  $R \rightarrow +\infty$  yields that

$$\iint_{\Omega} |\nabla \phi|^2 dx_1 dx_2 = 0.$$

Thus  $\nabla\phi = 0$  in  $\Omega$ . Since  $\phi = 0$  on  $\partial\Omega$ , one has  $\phi = 0$  in  $\Omega$ . This proves the uniqueness of subsonic solution to the problem (2.17).

## 6. EXISTENCE OF THE CRITICAL DENSITY IN THE UPSTREAM

In this section, we show that there exists a critical density  $\rho_{cr}$  such that there exists a subsonic solution as long as the density of the incoming flows is less than  $\rho_{cr}$ .

**Proposition 6.1.** *There exists a critical value  $\underline{\rho}_0 > 0$ , such that if  $\rho_0 > \underline{\rho}_0$ , there exists a unique  $\psi$  which solves the following problem*

$$\begin{cases} \operatorname{div} \left( \frac{\nabla\psi}{H(|\nabla\psi|^2, \psi; \rho_0)} \right) = F(\psi; \rho_0)F'(\psi; \rho_0)H & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma, \end{cases} \quad (6.1)$$

and satisfies

$$\psi \geq 0, \quad |\psi - \bar{\psi}| \leq C\rho_0 \quad \text{in } \bar{\Omega}, \quad \|\nabla(\psi - \bar{\psi})\|_{L^2(\Omega)} \leq \mathcal{C}\rho_0, \quad \text{and} \quad M(\rho_0) = \sup_{\bar{\Omega}} \frac{|\nabla\psi|}{\Sigma(\mathcal{B}(\psi; \rho_0))} < 1. \quad (6.2)$$

Moreover, either  $M(\rho_0) \rightarrow 1$  as  $\rho_0 \downarrow \underline{\rho}_0$  or there does not exist a  $\sigma > 0$  such that the problem (6.1) has a solution for all  $\rho_0 \in (\underline{\rho}_0 - \sigma, \underline{\rho}_0)$  and

$$\sup_{\rho_0 \in (\underline{\rho}_0 - \sigma, \underline{\rho}_0)} M(\rho_0) < 1. \quad (6.3)$$

*Proof.* The key idea of the proof is similar to the proof of [40, Proposition 6].

Let  $\{\varepsilon_n\}_{n=1}^\infty$  be a strictly decreasing positive sequence satisfying  $\varepsilon_1 \leq 1/4$  and  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , and  $\zeta_n$  be a sequence of smooth increasing odd functions satisfying

$$\zeta_n(z) = \begin{cases} z & \text{if } |z| < 1 - 2\varepsilon_n, \\ 1 - 3\varepsilon_n/2 & \text{if } z \geq 1 - \varepsilon_n. \end{cases} \quad (6.4)$$

For  $\check{\mathcal{B}}_L(\psi; \rho_0)$  defined in (3.12), set

$$\check{H}_L^{(n)}(|\nabla\psi|^2, \psi; \rho_0) = \mathcal{H} \left( \zeta_n^2 \left( \frac{|\nabla\psi|}{\Sigma(\check{\mathcal{B}}_L(\psi; \rho_0))} \right) \Sigma^2(\check{\mathcal{B}}_L(\psi; \rho_0)), \check{\mathcal{B}}_L(\psi; \rho_0) \right).$$

We first study the problem

$$\begin{cases} \left( \left( 1 - \zeta^2 \left( \frac{|\nabla\psi|}{\sqrt{\gamma}(\check{H}_L^{(n)})^{\frac{\gamma+1}{2}}} \right) \right) \delta_{ij} + \zeta \left( \frac{\partial_i\psi}{\sqrt{\gamma}(\check{H}_L^{(n)})^{\frac{\gamma+1}{2}}} \right) \zeta \left( \frac{\partial_j\psi}{\sqrt{\gamma}(\check{H}_L^{(n)})^{\frac{\gamma+1}{2}}} \right) \right) \partial_{ij}\psi \\ \quad = \frac{\check{W}_L(\check{H}_L^{(n)})^2}{1 - \zeta^2 \left( \frac{|\nabla\psi|}{\sqrt{\gamma}(\check{H}_L^{(n)})^{\frac{\gamma+1}{2}}} \right)} \quad \text{in } \Omega_L, \\ \psi = 0 \quad \text{on } \Gamma, \quad \psi = m_L \quad \text{on } \Gamma_L, \end{cases} \quad (6.5)$$

where  $m_L$  is defined in (3.7). Similar to the proof for Lemma 3.2, one can show that the problem (6.5) has a solution  $\psi_L$  for any  $\rho_0 > \rho_0^*$ . Given  $\rho_0 \in (\rho_0^*, \infty)$ , let  $\mathcal{S}_L^{(n)}(\rho_0)$  be the set of all solutions of the problem (6.5). Denote

$$\mathcal{M}_L^{(n)}(\rho_0) = \sup_{\psi_L \in \mathcal{S}_L(\rho_0)} \sup_{x \in \Omega_L} \frac{|\nabla \psi_L(x; \rho_0)|}{\Sigma(\mathcal{B}_L(\psi_L; \rho_0))} \quad (6.6)$$

and

$$\rho_{0,L}^{(n)} = \inf \left\{ s \mid \text{for any } \rho_0 \geq s, \mathcal{M}_L^{(n)}(\rho_0) \leq 1 - 3\varepsilon_n \right\}. \quad (6.7)$$

It is easy to see that  $\rho_{0,L}^{(n)} \leq \bar{\rho}_0$ . For any  $\rho_0 > \rho_{0,L}^{(n)}$ , Corollary 3.5 and Lemma 3.7 show that the associated subsonic solution  $\psi_L$  of the problem (6.5) satisfies

$$|\psi_L - \bar{\psi}_L| \leq C\rho_0 \quad \text{and} \quad \|\psi_L - \bar{\psi}_L\|_{L^2(\Omega_L)} \leq \mathcal{C}^{(n)}\rho_0, \quad (6.8)$$

where  $\mathcal{C}^{(n)}$  depends on  $\varepsilon_n$ . Define

$$\underline{\rho}_0^{(n)} = \liminf_{L \rightarrow \infty} \rho_{0,L}^{(n)}. \quad (6.9)$$

If  $\rho_0 > \underline{\rho}_0^{(n)}$ , there exists a solution  $\psi(\cdot, \rho_0)$  of the problem (2.17) satisfying

$$\sup_{\Omega} \frac{|\nabla \psi(\cdot; \rho_0)|}{\sqrt{\gamma} H^{\frac{\gamma+1}{2}} (|\nabla \psi|^2, \psi; \rho_0)} \leq 1 - 4\varepsilon_n.$$

Furthermore, it is easy to see that  $\{\underline{\rho}_0^{(n)}\}$  is a decreasing sequence. Define  $\underline{\rho}_0 = \inf \underline{\rho}_0^{(n)}$ . If  $\rho_0 > \underline{\rho}_0$ , there is always a solution  $\psi$  of the problem (6.1). The same argument in [40, Proposition 6] gives that  $\underline{\rho}_0$  is the critical value described in Proposition 6.1. This finishes the proof of the proposition.  $\square$

As a direct consequence of Proposition 6.1, we complete the proof of Theorem 1.1.

## 7. SUBSONIC EULER FLOWS WITH GENERAL INCOMING VELOCITY

In this section, we consider the existence and uniqueness of subsonic Euler flows past a wall when the incoming horizontal velocities are general small perturbations of a constant.

**Proof of Theorem 1.2.** Theorem 1.2 is proved in a similar fashion as that for Theorem 1.1, so we only sketch the proof and emphasize on the main differences as follows. The main difference is that the convex condition  $u_0''(x_2) \geq 0$  in Theorem 1.1 is replaced by the smallness of  $u_0'(x_2)$  in (1.17).

**Step 1. Subsonic solutions in nozzles and their uniform estimates.** For  $L > 0$  sufficiently large, let  $g_L$  be defined in (3.4) and  $u_{0,L}(x_2) = u_0(0) + \int_0^{x_2} g_L(s) ds$ . Hence for  $x_2 \in [0, L]$ , we have

$$u_{0,L}(x_2) \geq \inf_{x_2 \in [0, L-1]} u_0(x_2) - \frac{|u_0'(L-1)|}{2}.$$

If  $L$  is sufficiently large, then  $u_{0,L}(x_2) \geq \bar{u}/2$  for all  $x_2 \in [0, L]$ . Furthermore,  $u_{0,L}(x_2)$  satisfies  $u_{0,L}'(L) = g_L(L) = 0$ .

Let  $F_L(\psi; \rho_0)$ ,  $W_L(\psi; \rho_0)$ ,  $m_L$  be the same as that in (3.7). We have the following proposition.

**Proposition 7.1.** *For any  $k > 1$ , there exists a constant  $\varepsilon_1 > 0$  independent of  $L$  such that if  $u_0(x_2)$  satisfies the conditions (1.16)-(1.17) with  $\varepsilon \in (0, \varepsilon_1)$ , then there exists a  $\bar{\rho}_0 \in (\rho_0^*, \infty)$  independent of  $L$ , such that if  $\rho_0 > \bar{\rho}_0$ , there exists a solution  $\psi_L \in C^{2,\alpha}(\Omega_L) \cap C^{1,\beta}(\bar{\Omega}_L)$  of the problem (3.9) and satisfies*

$$0 \leq \psi_L \leq m_L \quad \text{in } \Omega_L \quad \text{and} \quad \sup_{x \in \Omega_L} \frac{|\nabla \psi_L|}{\Sigma(\mathcal{B}_L(\psi_L; \rho_0))} \leq \frac{1}{4}. \quad (7.1)$$

*Proof.* First, let  $\tilde{F}_L$  be the same as that in (3.11). Define

$$\check{\mathfrak{B}}_L(\Psi, x; \rho_0) = h(\rho_0) + \frac{\tilde{F}_L^2(\bar{\psi}_L + \bar{S}\zeta(\frac{\Psi}{\bar{S}}); \rho_0)}{2} \quad (7.2)$$

and

$$\check{\mathfrak{M}}_L(\Psi, x; \rho_0) = \tilde{F}_L\left(\bar{\psi}_L + \bar{S}\zeta\left(\frac{\Psi}{\bar{S}}\right); \rho_0\right) \tilde{F}_L'\left(\bar{\psi}_L + \bar{S}\zeta\left(\frac{\Psi}{\bar{S}}\right); \rho_0\right) \quad (7.3)$$

where  $\zeta$  and  $\bar{\psi}_L$  are defined in (3.13) and (3.22), respectively, and

$$\bar{S} = 2\rho_0(\bar{U} + 1) \quad \text{and} \quad \bar{U} = \sup_{x_1 \in \mathbb{R}} \int_0^{f(x_1)} u_0(s) ds.$$

If  $x_2 \geq \frac{2\bar{S}}{\rho_0 \min u_{0,L}}$ , then

$$\frac{\rho_0 \min u_{0,L}}{2} x_2 \leq \bar{\psi}_L + \bar{S}\zeta\left(\frac{\Psi}{\bar{S}}\right) \leq 2\rho_0 \max u_{0,L} x_2. \quad (7.4)$$

If  $x_2 \in \left(0, \frac{2\bar{S}}{\rho_0 \min u_{0,L}}\right)$ , then

$$\bar{\psi}_L + \bar{S}\zeta\left(\frac{\Psi}{\bar{S}}\right) \geq -\bar{S}. \quad (7.5)$$

Therefore, it follows from (1.17) that there exists a uniform constant  $C$  such that

$$|\check{\mathfrak{M}}_L(\Psi, x; \rho_0)| \leq \frac{C\varepsilon}{\rho_0(1+x_2)^{k+1}}. \quad (7.6)$$

Set  $\check{\mathfrak{H}}_L(\nabla \Psi, \Psi, x; \rho_0) = \mathcal{H}\left(\zeta^2\left(\frac{|\nabla(\bar{\psi}_L + \Psi)|}{\Sigma(\check{\mathfrak{B}}_L(\Psi, x; \rho_0))}\right) \Sigma^2(\check{\mathfrak{B}}_L(\Psi, x; \rho_0)), \check{\mathfrak{B}}_L(\Psi; \rho_0)\right)$  and  $\Omega_{L,N} = \Omega_L \cap \{x \mid |x_1| \leq N\}$ . We first study the problem

$$\begin{cases} \mathfrak{A}_{L,ij}(\nabla \Psi, \Psi, x; \rho_0) \partial_{ij} \Psi = Q_L(\nabla \Psi, \Psi, x; \rho_0) & \text{in } \Omega_{L,N}, \\ \Psi = -\rho_0 \int_0^{f(x_1)} u_{0,L}(s) ds & \text{on } \Gamma \cap \partial\Omega_{L,N}, \quad \Psi = 0 & \text{on } \partial\Omega_{L,N} \setminus \Gamma, \end{cases} \quad (7.7)$$



where

$$\begin{aligned} \mathfrak{A}_{L;ij}(\nabla\Psi, \Psi, x; \rho_0) &= \left(1 - \zeta^2 \left( \frac{|\nabla(\Psi_L + \bar{\psi}_L)|}{\sqrt{\gamma}\check{\mathfrak{H}}_L^{\frac{\gamma+1}{2}}(\nabla\Psi, \Psi, x; \rho_0)} \right) \right) \delta_{ij} \\ &+ \zeta \left( \frac{\partial_i(\Psi_L + \bar{\psi}_L)}{\sqrt{\gamma}\check{\mathfrak{H}}_L^{\frac{\gamma+1}{2}}(\nabla\Psi, \Psi, x; \rho_0)} \right) \zeta \left( \frac{\partial_j(\Psi_L + \bar{\psi}_L)}{\sqrt{\gamma}\check{\mathfrak{H}}_L^{\frac{\gamma+1}{2}}(\nabla\Psi, \Psi, x; \rho_0)} \right) \end{aligned} \quad (7.8)$$

and

$$Q_L(\nabla\Psi, \Psi, x; \rho_0) = \check{\mathfrak{M}}_L(\Psi, x; \rho_0) \left( \check{\mathfrak{H}}_L^2(\nabla\Psi, \Psi, x; \rho_0) - \rho_0^2 - \rho_0^2 \zeta^2 \left( \frac{|\partial_1\Psi|}{\sqrt{\gamma}\check{\mathfrak{H}}_L^{\frac{\gamma+1}{2}}(\nabla\Psi, \Psi, x; \rho_0)} \right) \right).$$

It follows from (7.6) that  $Q_L$  satisfies

$$|Q_L(\nabla\Psi, \Psi, x; \rho_0)| \leq \frac{C_1 \rho_0 \varepsilon}{(1+x_2)^{k+1}}, \quad (7.9)$$

where  $C_1$  is independent of  $L$ . The eigenvalues for the matrix  $A_L$  are

$$\lambda_L = 1 - \sum_{i=1}^2 \zeta^2 \left( \frac{|\partial_i(\Psi_L + \bar{\psi}_L)|}{\sqrt{\gamma}\check{\mathfrak{H}}_L^{\frac{\gamma+1}{2}}(\nabla\Psi, \Psi, x; \rho_0)} \right) \quad (7.10)$$

and

$$\Lambda_L = 1 - \sum_{i=1}^2 \zeta^2 \left( \frac{|\partial_i(\Psi_L + \bar{\psi}_L)|}{\sqrt{\gamma}\check{\mathfrak{H}}_L^{\frac{\gamma+1}{2}}(\nabla\Psi, \Psi, x; \rho_0)} \right) + \zeta^2 \left( \frac{|\nabla(\Psi_L + \bar{\psi}_L)|}{\sqrt{\gamma}\check{\mathfrak{H}}_L^{\frac{\gamma+1}{2}}(\nabla\Psi, \Psi, x; \rho_0)} \right). \quad (7.11)$$

As usual,  $\Lambda_L/\lambda_L$  is called the elliptic ratio for the equation in (7.7).

Let  $\hat{\phi} = \frac{\varepsilon \rho_0}{(1+x_2)^{k-1}}$ . Obviously, one has  $\partial_{x_1}(\bar{\psi} + C\hat{\phi}) = 0$  for any constant  $C$ . Hence direct computations give

$$\mathfrak{A}_{L;ij}(\nabla(C_2\hat{\phi}), \Psi, x; \rho_0) \partial_{ij}(C_2\hat{\phi}) = C_2 \partial_{x_2 x_2} \left( \frac{\varepsilon \rho_0}{(1+x_2)^{k-1}} \right) = \frac{C_2 k(k-1) \rho_0 \varepsilon}{(1+x_2)^{k+1}}.$$

Therefore, choosing  $C_2$  sufficiently large yields that

$$\begin{cases} \mathfrak{A}_{L;ij}(\nabla(C_2\hat{\phi}), \Psi, x; \rho_0) \partial_{ij}(C_2\hat{\phi}) \geq Q_L = \mathfrak{A}_{L;ij}(\nabla\Psi, \Psi, x; \rho_0) \partial_{ij}\Psi_{L,N} & \text{in } \Omega_{L,N}, \\ C_2\hat{\phi} \geq 0 \geq -\rho_0 \int_0^{f(x_1)} u_{0,L}(s) ds = \Psi_{L,N} & \text{on } \Gamma \cap \partial\Omega_{L,N}, \\ C_2\hat{\phi} \geq 0 = \Psi_{L,N} & \text{on } \partial\Omega_{L,N} \setminus \Gamma. \end{cases}$$

Therefore, the comparison principle for nonlinear elliptic equations [23, Theorem 10.1] implies that

$$\Psi_{L,N} \leq C_2 \hat{\phi} \leq C_2 \frac{\rho_0 \varepsilon}{(1+x_2)^{k-1}} \quad \text{in } \Omega_L.$$

One of our key observation is that the constant  $C_2$  depends neither on  $L$  nor on the elliptic coefficients. Note also that  $C_2$  does not depend on the elliptic ratio  $\Lambda_L/\lambda_L$ .

Similarly, one has

$$\Psi_{L,N} \geq -\rho_0 \bar{U} - C_2 \frac{\rho_0 \varepsilon}{(1+x_2)^{k-1}}.$$

Therefore, choosing  $\varepsilon \leq \frac{1}{C_2}$  yields that for sufficiently large  $L$ , the following estimate holds

$$-\bar{U} - \rho_0 < \Psi_{L,N} < \rho_0 \quad \text{in } \Omega_L.$$

Thus we have the following uniform  $L^\infty$ -norm estimate

$$|\Psi_{L,N}| \leq \rho_0 (\bar{U} + 1). \quad (7.12)$$

Next, similar to the proof for Lemma 3.6 in Subsection 3.2, it follows from the Hölder gradient estimate [23, Theorem 12.4] and the estimate near the corners [15, Section 2.2] for the elliptic equations that one has the following global estimate

$$\|\Psi_{L,2N}\|_{1,\beta;\Omega_{L,N}} \leq \mathcal{C} \rho_0, \quad (7.13)$$

where  $\mathcal{C}$  depends on the elliptic coefficients but is independent of  $L$ . Taking limit for  $N \rightarrow \infty$ , one gets that there exists a subsequence of  $\{\Psi_{L,N}\}$  converging to  $\Psi_L$  which satisfies

$$|\Psi_L| \leq \rho_0 (\bar{U} + 1) \quad \text{and} \quad \|\Psi_L\|_{1,\beta;\Omega_L} \leq \mathcal{C} \rho_0. \quad (7.14)$$

If  $\rho_0$  is sufficiently large, then

$$\frac{|\nabla(\Psi_L + \bar{\psi}_L)|}{\Sigma(\mathfrak{B}_L)} \leq \frac{|\nabla \Psi_L| + |\nabla \bar{\psi}_L|}{\Sigma(\mathfrak{B}_L)} \leq \mathcal{C}^\# \rho_0^{\frac{1-\gamma}{2}} \leq \frac{1}{4}. \quad (7.15)$$

Therefore,  $\psi_L = \bar{\psi}_L + \Psi_L$  solves the problem (3.9). Since  $\check{W}_L \leq 0$  if  $\psi_L \geq m_L$  and  $\check{W}_L \geq 0$  if  $\psi_L \leq 0$ , it follows from the maximum principle that

$$0 \leq \psi_L \leq m_L. \quad (7.16)$$

We now choose  $\varepsilon_1 = \frac{1}{C_2}$ , then the proposition is proved.  $\square$

Furthermore, we also have the following uniform integral estimate.

**Lemma 7.2.** *For any  $k > 1$ , there exists a constant  $\varepsilon_0 > 0$  independent of  $L$  such that if  $u_0(x_2)$  satisfies the conditions (1.16)-(1.17) with  $\varepsilon \in (0, \varepsilon_0)$ , and  $\psi_L$  is a subsonic solution of the problem (3.9), then we have*

$$|(\psi_L - \bar{\psi}_L)(x_1, x_2)| \rightarrow 0 \quad \text{uniformly with respect to } x_2 \text{ as } |x_1| \rightarrow \infty \quad (7.17)$$

and

$$\|\nabla(\psi_L - \bar{\psi}_L)\|_{L^2(\Omega_L)} \leq \mathcal{C}, \quad (7.18)$$

where the constant  $\mathcal{C}$  depends on the elliptic coefficients and is independent of  $L$ .

*Proof.* The far field behavior (7.17) follows from [40, Proposition 4].

In fact, it follows from (3.49) that

$$\begin{aligned}
& \iint_{\Omega_{L,N}} (a_{ij} \partial_i \Psi_L \partial_j \Psi_L + 2b_i \Psi_L \partial_i \Psi_L + d_1 \Psi_L^2) dx_1 dx_2 \\
& \leq \mathcal{C}(\rho_0) \left( \int_{x_1 \in \mathbb{R}} |\bar{\psi}_L(f(x_1))| dx_1 + \int_{f(N)}^L |\Psi_L(N, x_2)| dx_2 + \int_{f(-N)}^L |\Psi_L(-N, x_2)| dx_2 \right) \\
& \quad - \iint_{\Omega_{L,N}} d_2 \Psi_L^2 dx_1 dx_2.
\end{aligned} \tag{7.19}$$

Although the sign of the integral term  $d_2$  is not clear without the assumption  $u''_{0,L}(x_2) \geq 0$ , the following estimate holds

$$\begin{aligned}
|d_2(\nabla \psi_L, \psi_L; \nabla \bar{\psi}_L, \bar{\psi}_L)| &= \left| \int_0^1 \frac{H(|\nabla \bar{\psi}_{L,t}|^2, \bar{\psi}_{L,t}) u''_{0,L}(\kappa(\bar{\psi}_{L,t}; \rho_0))}{\rho_0^2 u_{0,L}(\kappa_L(\bar{\psi}_{L,t}; \rho_0))} dt \right| \\
&\leq \frac{C\varepsilon}{\rho_0 (1 + \kappa_L(\bar{\psi}_{L,t}; \rho_0))^{k+2}}.
\end{aligned} \tag{7.20}$$

Since

$$|\Psi_L| = |\psi_L - \bar{\psi}_L| = \left| \int_0^{\kappa_L(\psi_L; \rho_0) - x_2} \rho_0 u_{0,L}(s) ds \right| = \rho_0 \max u_{0,L} |\kappa_L(\psi_L; \rho_0) - x_2|,$$

it follows from (7.14) that

$$|\kappa_L(\psi_L; \rho_0) - x_2| \leq \frac{\bar{U} + 1}{\max u_0}.$$

Therefore,

$$-\frac{\bar{U} + 1}{\max u_0} + x_2 \leq \kappa(\bar{\psi}_{L,t}; \rho_0) \leq \frac{\bar{U} + 1}{\max u_0} + x_2. \tag{7.21}$$

Thus

$$|u''_{0,L}(\kappa(\bar{\psi}_{L,t}; \rho_0))| \leq \frac{C\varepsilon}{(1 + x_2)^{k+2}}. \tag{7.22}$$

This, together with (7.20) implies that

$$|d_2(\nabla \psi_L, \psi_L; \nabla \bar{\psi}_L, \bar{\psi}_L)| \leq \frac{C\varepsilon}{\rho_0 (1 + x_2)^{k+2}}. \tag{7.23}$$

Therefore, combining (7.23), (3.49), and (3.50) together yields that for any  $N > 0$ , one has

$$\begin{aligned}
& \iint_{\Omega_{L,N}} |\nabla \Psi_L|^2 dx_1 dx_2 \\
& \leq \mathcal{C}(\rho_0) \left( \int_{\mathbb{R}} |\bar{\Psi}_L(f(x_1))| dx_1 + \int_{f(N)}^L |\Psi_L(N, x_2)| dx_2 + \int_{f(-N)}^L |\Psi_L(-N, x_2)| dx_2 \right) \\
& \quad + C \iint_{\Omega_{L,N}} \frac{\varepsilon \Psi_L^2}{(1 + x_2)^{k+2}} dx_1 dx_2.
\end{aligned}$$

Noting that  $k > 1$  so that we apply the weighted Poincaré inequality in Lemma A.1 in Appendix for the last term in (7.25) to get

$$\begin{aligned} & \iint_{\Omega_{L,N}} |\nabla \Psi_L|^2 dx_1 dx_2 \\ & \leq \mathcal{C}(\rho_0) \left( \int_{\mathbb{R}} |\bar{\psi}_L(x_1, f(x_1))| dx_1 + \int_{f(N)}^L |\Psi_L(N, x_2)| dx_2 + \int_{f(-N)}^L |\Psi_L(-N, x_2)| dx_2 \right) \\ & \quad + \frac{C}{k+1} \int_{\mathbb{R}} |\Psi_L(x_1, f(x_1))|^2 dx_1 + \frac{4C_3\varepsilon}{(k+1)^2} \iint_{\Omega_{L,N}} |\nabla \Psi_L|^2 dx_1 dx_2. \end{aligned}$$

Choosing  $\varepsilon_0 \leq \min \left( \varepsilon_1, \frac{(k+1)^2}{8C_3} \right)$  yields

$$\begin{aligned} & \iint_{\Omega_L} |\nabla \Psi_L|^2 dx_1 dx_2 \\ & \leq \mathcal{C}(\rho_0) \left( \int_{\mathbb{R}} |\bar{\psi}_L(x_1, f(x_1))| dx_1 + \int_{f(N)}^L |\Psi_L(N, x_2)| dx_2 + \int_{f(-N)}^L |\Psi_L(-N, x_2)| dx_2 \right) \quad (7.24) \\ & \quad + \frac{2}{k+1} \int_{\mathbb{R}} |\Psi_L(x_1, f(x_1))|^2 dx_1. \end{aligned}$$

Taking limit  $N \rightarrow \infty$  and using (7.17) show

$$\int_{\Omega_L} |\nabla \Psi_L|^2 dx_1 dx_2 \leq \mathcal{C}(\rho_0) \int_{\mathbb{R}} |\bar{\psi}_L(f(x_1))| dx_1 + \frac{C}{k+1} \int_{\mathbb{R}} |\Psi_L(x_1, f(x_1))|^2 dx_1. \quad (7.25)$$

This finishes the proof of the lemma.  $\square$

**Step 2. Existence of subsonic solutions and their fine properties.** The subsonic solution  $\Psi$  on  $\Omega$  is then obtained as a limit of  $\{\Psi_L\}$ .

**Proposition 7.3.** *Suppose that  $u_0(x_2)$  satisfies the conditions (1.16)-(1.17), then there exists a  $\bar{\rho}_0 \geq \rho_0^*$ , such that if  $\rho_0 > \bar{\rho}_0$ , there exists a solution  $\Psi \in C^{2,\alpha}(\Omega) \cap C^{1,\beta}(\bar{\Omega})$  satisfying*

$$\Psi + \bar{\psi} \geq 0 \quad \text{and} \quad |\nabla(\Psi + \bar{\psi})| < \frac{1}{4} \Sigma(\Psi + \bar{\psi}; \rho_0). \quad (7.26)$$

*Proof.* Taking the limit  $L \rightarrow \infty$  gives

$$|\nabla(\bar{\psi} + \Psi)| \leq C\rho_0. \quad (7.27)$$

Hence, if  $\rho_0$  is sufficiently large, one has

$$\frac{|\nabla(\bar{\psi} + \Psi)|}{\Sigma(\bar{\psi} + \Psi)} \leq 1/4. \quad (7.28)$$

This shows that the subsonic truncation can be removed. So the proof of the proposition is finished.  $\square$

Furthermore, it follows from the estimate (7.18) that

$$\|\nabla \Psi\|_{L^2(\mathbb{R}_+^2)} \leq C. \quad (7.29)$$

Combining (7.29) with the Hölder gradient estimate yields the asymptotic behavior of the flows. Using the same idea in Subsection 4.2, one can prove that the flows in  $\Omega$  have positive horizontal velocity except at the corner points.

**Step 3. The uniqueness of solution.** In order to prove the uniqueness, we also study the problem for the difference of two solutions. The same arguments in Section 5 give (5.5). Now the key task is to estimate the following term

$$\iint_{\Omega} \eta^2 \int_0^1 H(|\nabla \psi_{(1+t)}|^2, \psi_{(1+t)}) \phi^2 (F(\psi_{(1+t)}) F''(\psi_{(1+t)}) + (F'(\psi_{(1+t)}))^2) dt dx_1 dx_2,$$

where  $\eta$  is the smooth cut-off function with (5.4),  $\phi$  is the difference of the two subsonic solutions  $\psi_1$  and  $\psi_2$ , and  $\psi_{(1+t)} = (1-t)\psi_1 + t\psi_2$  for  $t \in [0, 1]$ .

Note that

$$\left| F(\psi_{(1+t)}) F''(\psi_{(1+t)}) + (F'(\psi_{(1+t)}))^2 \right| = \left| \frac{u_0''(\kappa(\psi_{(1+t)}; \rho_0))}{\rho_0^2 u_0(\kappa(\psi_{(1+t)}; \rho_0))} \right|.$$

The same argument as for (7.22) gives

$$u_0''(\kappa(\psi_{(1+t)}; \rho_0)) \leq \frac{C\varepsilon}{\rho_0^2(1+x_2)^{k+2}}, \quad (7.30)$$

where  $C$  does not depend on  $L$  or elliptic coefficients. Therefore,

$$\begin{aligned} & \iint_{B_{2R}^{\Omega}(0)} \eta^2 \int_0^1 H(|\nabla \psi_{(1+t)}|^2, \psi_{(1+t)}) \phi^2 (F(\psi_{(1+t)}) F''(\psi_{(1+t)}) + (F'(\psi_{(1+t)}))^2) dt dx_1 dx_2 \\ & \leq C\varepsilon \iint_{\Omega} \frac{\eta^2 \phi^2}{(1+x_2)^{k+2}} dx_1 dx_2. \end{aligned} \quad (7.31)$$

Noting that  $\phi = 0$  on  $\Gamma$  and using the weighted Poincaré inequality in Lemma A.1 yield

$$\begin{aligned} & \iint_{B_{2R}^{\Omega}(0)} \eta^2 \int_0^1 H(|\nabla \psi_{(1+t)}|^2, \psi_{(1+t)}) \phi^2 (F(\psi_{(1+t)}) F''(\psi_{(1+t)}) + (F'(\psi_{(1+t)}))^2) dt dx_1 dx_2 \\ & \leq C\varepsilon \iint_{\Omega} \eta^2 |\nabla \phi|^2 + |\nabla \eta|^2 \phi^2 dx_1 dx_2 \\ & \leq C\varepsilon \iint_{B_{2R}^{\Omega}(0)} \eta^2 |\nabla \phi|^2 dx_1 dx_2 + C\varepsilon \iint_{A_{R,2R}^{\Omega}(0)} |\nabla \eta|^2 dx_1 dx_2. \end{aligned} \quad (7.32)$$

Hence, substituting (7.32) into (5.5) and applying the same technique in (5.7) and (5.8) give that

$$\begin{aligned}
& \iint_{B_{2R}^\Omega(0)} \int_0^1 |F(\psi_{(1+t)})F'(\psi_{(1+t)})\eta\phi|^2 dt dx_1 dx_2 \\
& \leq \mathcal{C} \iint_{B_{2R}^\Omega(0)} |\nabla\phi|^2 dx_1 dx_2 + \mathcal{C}\delta \iint_{B_{2R}^\Omega(0)} \eta^2 \left( \int_0^1 F(\psi_{(1+t)})F'(\psi_{(1+t)})\phi dt \right)^2 dx_1 dx_2 \\
& \quad + \mathcal{C}(\delta) \iint_{A_{R,2R}^\Omega(0)} |\nabla\eta|^2 dx_1 dx_2,
\end{aligned} \tag{7.33}$$

Choosing suitable small  $\delta > 0$  yields

$$\begin{aligned}
& \iint_{B_R^\Omega(0)} \int_0^1 |F(\psi_{(1+t)})F'(\psi_{(1+t)})\phi|^2 ds dx_1 dx_2 \\
& \leq \mathcal{C} \iint_{B_{2R}^\Omega(0)} |\nabla\phi|^2 dx_1 dx_2 + \mathcal{C}(\delta) \iint_{A_{R,2R}^\Omega(0)} |\nabla\eta|^2 dx_1 dx_2.
\end{aligned} \tag{7.34}$$

Hence we obtain the key uniform estimate

$$\left\| \phi \int_0^1 |F(\psi_{(1+t)})F'(\psi_{(1+t)})| dt \right\|_{L^2(\Omega)} \leq \mathcal{C}.$$

This is exactly the same as (5.9). Similarly, one can prove (5.10) which implies  $\phi = 0$  in  $\Omega$ . Hence the uniqueness for subsonic solution is proved.

**Step 4. Existence of critical value  $\rho_{cr}$  for the density in the upstream.** Note that the choice of  $\varepsilon_1$  and  $\varepsilon_0$  does not depend on the elliptic coefficients, so the proof for the existence of critical value for the incoming density in the upstream is similar to the one in Section 6.  $\square$

## 8. LIMIT OF SUBSONIC FLOWS

In this section, we prove Theorem 1.4. Given a sequence of  $\{M_0^{(n)}\}$  converging to  $\rho_{cr}$ , the associated subsonic flows  $\{(\rho_n, u_n, v_n)\}$  satisfy the Euler system (1.1) and the following three conditions

- (1) the Mach number of the flows  $\frac{\sqrt{u_n^2 + v_n^2}}{\sqrt{\gamma\rho_n^{\gamma-1}}} \leq 1$  a.e. in  $\Omega$ ;
- (2) the Bernoulli functions of the flows  $\frac{u_n^2 + v_n^2}{2} + h(\rho_n)$  are uniformly bounded above and below;
- (3) the vorticities of the flows  $\partial_{x_2}u_n - \partial_{x_1}v_n$  are uniformly bounded measures.

Hence, using Theorem 2.2 in [8] yields that there exists a subsequence still labelled by  $\{(\rho_n, u_n, v_n)\}$  converging to  $(\rho, u, v)$  a.e. in  $\Omega$ . Thus  $(\rho, u, v)$  also solves the Euler system (1.1) in the weak sense and the boundary condition (1.4) in the sense of normal trace. This finishes the proof of Theorem 1.4.

## APPENDIX A. THE WEIGHTED POINCARÉ INEQUALITY

In this appendix, we give a weighted Poincaré inequality and its proof, which is used in Section 7.

**Lemma A.1.** *Let  $I = (a, b)$  with  $a \geq 0$  and  $b$  could be infinity. If  $g = g(s) \in H^1(I)$ , then for any  $l > 2$ , it holds that*

$$\int_I \frac{g^2(s)}{(s+1)^l} ds \leq \frac{2g^2(a)}{l-1} + \frac{4}{(l-1)^2} \int_I (g'(s))^2 ds. \quad (\text{A.1})$$

*Proof.* Integration by parts and using the Cauchy-Schwartz inequality give

$$\begin{aligned} & \int_I \frac{g^2(s)}{(1+s)^l} ds \\ &= \frac{1}{1-l} \left[ \frac{g^2(s)}{(1+s)^{l-1}} \Big|_{s=a}^{s=b} - 2 \int_I g(s)g'(s)(1+s)^{1-l} ds \right] \\ &\leq \frac{1}{l-1} \left[ \frac{g^2(a)}{(1+a)^{l-1}} - \frac{g^2(b)}{(1+b)^{l-1}} \right] + \frac{1}{l-1} \left[ c \int_I \frac{g^2(s)}{(1+s)^l} ds + \frac{1}{c} \int_I \frac{(g'(s))^2}{(1+s)^{l-2}} ds \right]. \end{aligned}$$

Hence,

$$(l-1-c) \int_I \frac{g^2(s)}{(1+s)^l} ds \leq g^2(a) + \frac{1}{c} \int_I \frac{(g'(s))^2}{(1+s)^{l-2}} ds.$$

Taking  $c = \frac{l-1}{2}$  and noting that  $l > 2$  yield

$$\int_I \frac{g^2(s)}{(1+s)^l} ds \leq \frac{2}{l-1} g^2(a) + \frac{4}{(l-1)^2} \int_I (g'(s))^2 ds.$$

This finishes the proof of the inequality.  $\square$

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